Math 725 Problem set I.

1. Let $g_i$ be measurable nonnegative functions on a measure space $X$, with measure $\mu$, and let $r \geq 1$. Then prove the inequality
\[
\left( \int_X \prod_{i=1}^n |g_i|^r \,d\mu \right)^{1/r} \leq \prod_{i=1}^n \left( \int |g_i|^{q_i} \,d\mu \right)^{1/q_i},
\]
where $\sum_{i=1}^n \frac{1}{q_i} = \frac{1}{r}$.

2. In $\mathbb{R}^n$ let, for $j = 1, \ldots, n$, $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projections
\[
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\]
Let $g_j$, $j = 1, \ldots, n$ be functions in $L^{n-1}(\mathbb{R}^{n-1})$.
Prove the inequality
\[
\int_{\mathbb{R}^n} \prod_{j=1}^n g_j(\pi_j(x)) \,dx \leq \prod_{j=1}^n \|g_j\|_{L^{n-1}(\mathbb{R}^{n-1})}.
\]
*Hint:* It may help to consider the cases $n = 2, 3$ first.

3. Let $X$, $Y$ be measure spaces with positive measures $d\mu$, $d\nu$, respectively, and let $K$ be a measurable function on $X \times Y$. Let $u$, $w$ be nonnegative measurable functions on $X$, $Y$ respectively, and assume that $u(x) > 0$ a.e. and $w(y) > 0$ a.e. Suppose $1 < p < \infty$ and
\[
\int_X |K(x,y)|u(x)d\mu \leq Bw(y), \quad \nu - \text{a.e.}
\]
\[
\int_Y |K(x,y)|w(y)^{\frac{1}{p-1}}d\nu \leq Au(x)^{\frac{1}{p-1}}, \quad \mu - \text{a.e.}
\]
Let $T$ be defined by $Tf(x) = \int_Y K(x,y)f(y)d\nu$. Show that $T$ maps $L^p(Y)$ to $L^p(X)$ and the operator norm is bounded by $A^{1-1/p}B^{1/p}$.
*Remark:* Note that the two conditions are similar when $p = 2$.

4. Let
\[
a(x,y) = \begin{cases} 
1 & \text{if } |x-y| \geq |x| \\
0 & \text{if } |x-y| < |x|
\end{cases}
\]
and let
\[
Tf(x) = \int \frac{a(x,y)}{|x-y|}f(y)dy.
\]
Show that $T$ is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

5. Folland No. 34 on p. 197.

6. A separable space is a normed space which contains a dense countable subset. Show that $L^p(\mathbb{R}^n)$ is separable for $1 \leq p < \infty$ but $L^\infty(\mathbb{R}^n)$ is not separable.

7. Folland No. 40, p. 199.