1. Let $f \in L^1(\mathbb{R}^d)$ so that $f * f = f$. Find $f$.

2. For $f \in \mathcal{S}(\mathbb{R}^d)$ let $T f(\xi) = (2\pi)^{-d/2} \widehat{f}(\xi)$.

(i) Show that $T^4$ is the identity operator and every $f \in \mathcal{S}(\mathbb{R}^d)$ has a unique decomposition $f = f_0 + f_1 + f_2 + f_3$ so that $f_k \in \mathcal{S}(\mathbb{R}^n)$ and $T f_k = i^k f_k$ for $k = 0, 1, 2, 3$.

(ii) Define $L_j f(x) = x_j f(x) + \partial f / \partial x_j$, for $j = 1, \ldots, d$.

Show that $L_j : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ is surjective. What is the kernel (nullspace) of the linear map $L_j$? With $T, f_k$ as in (i) show that $T L_j f_k = i^{k+1} L_j f_k$, for $j = 1, \ldots, d$.

3.

(i) Let

$$S_R f(x) = \frac{1}{2\pi} \int_{-R}^{R} \widehat{f}(\xi) e^{i(x,\xi)} \, d\xi$$

be the “partial sum operator”. If $f \in L^2$ prove that

$$\lim_{R \to \infty} \| S_R f - f \|_2 = 0.$$ 

(ii) The Fejér kernels $K_t$ on the real line are given by

$$K_t(\xi) = \max \{ 1 - |\xi|/t, 0 \}.$$ 

Compute $K_t(x)$. Show that for the “arithmetic means” of the $S_R$ we have

$$\mathcal{S}_t(f) := \frac{1}{2t} \int_{-t}^{t} S_R f \, dR = K_t * f.$$ 

(iv) Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{t \to \infty} \| \mathcal{S}_t(f) - f \|_1 = 0.$$ 

Formulate generalizations using problem 9e/f below.

(v) Let $f \in L^1(\mathbb{R})$ and assume $\widehat{f}(\xi) \geq 0$. Suppose that $f$ is continuous at 0. Show that $\widehat{f} \in L^1(\mathbb{R})$ and $f(0) = \frac{1}{2\pi} \int \widehat{f}(\xi) \, d\xi$.

**Hint:** If $\{G_n\}$ is a monotone numerical sequence then the existence of the limit of arithmetic means, $\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} G_k$, implies the existence of $\lim_{n \to \infty} G_n$ (the converse is of course true without the monotonicity assumption).

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1Recall that $\widehat{f}(\xi) = \int f(y) e^{-i(y,\xi)} \, dx$. 
4. Compute the Fourier transform of the following functions defined on $\mathbb{R}$.
   (i) $(1 + x^2)^{-1}$
   (ii) $(1 + (x - a)^2)^{-2}$
   (iii) $e^{-|x|}$
   (iv) $\frac{1}{P(x)}$ where $P$ is a polynomial of degree $\geq 2$ and no roots lie on the real axis.
   (v) $\chi_{[-a,a]}(x)$.
   (vi) $\sin x$
   (vii) $|x|^a$ with $-1 < \Re(a) < 0$
   (viii) $|x + 2|^{-1/2}$
   (ix) $x \sin x$

5. The De-la-Vallée-Poussin kernels $V_t$ are given by
   $$
   V_t(\xi) = \begin{cases}
   1, & |\xi| \leq t \\
   2 - \frac{|\xi|}{t}, & t < |\xi| \leq 2t \\
   0, & |\xi| > 2t
   \end{cases}
   $$

Show that $V_t$ is an approximate identity, i.e. it satisfies $\int V_t(x)dx = 1$, and
   $$
   \lim_{t \to \infty} \int_{|x| > t} |V_t(x)|dx \to 0 \text{ as } t \to \infty.
   $$
   Thus, if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ we have $V_t * f \to f$ in the $L^p$ norm and almost everywhere; i.e.
   $$
   \lim_{t \to \infty} \frac{1}{2\pi} \int V_t(\xi)e^{i(x,\xi)}d\xi = f(x)
   $$

in $L^p$ and almost everywhere.

**Hint:** Show the connection with the Fejér kernel: $V_t = 2K_{2t} - K_t$.

6. Let $\phi \in S(\mathbb{R}^n)$, so that $\int \phi = 1$ and set $\phi_\varepsilon = \varepsilon^{-n}\phi(x/\varepsilon)$.
   (i) Show: If $f \in S$ then $f * \phi_\varepsilon$ converges to $f$ in the topology of $S$.
   (ii) Show: If $u \in S'$ then $u * \phi_\varepsilon$ converges to $u$ in the (weak-*) topology of $S'$.

7. (i) Let $g(x) = e^x \cos e^x$. Show that there is a tempered distribution $\Lambda_g$ so that
   $$
   \langle \Lambda_g, \phi \rangle = \int g(x)\phi(x)dx \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n).
   $$
   (ii) Let $h(x) = e^x$. Show that there is no tempered distribution $u$ so that
   $$
   \langle u, \phi \rangle = \int h(x)\phi(x)dx \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n).
   $$

8. Let $K \in S'$ and $f \in S$. Prove that $\widehat{K * f} = \hat{K}f$ and derive a formula for $\hat{K}f$. 
9. Prove that the following functions are Fourier transforms of an $L^1(\mathbb{R}^n)$ function.
   
a) $m(\xi) = (1 + |\xi|^2)^{-\epsilon}$, $\epsilon > 0$.
   
b) $m(\xi) = e^{i\xi \cdot a}(1 + |\xi|^2)^{-\epsilon}$ where $a \in \mathbb{R}^n$ and $\epsilon > 0$.
   
c) $m(\xi) = e^{i\xi \cdot a}\chi_{\infty}(\xi)|\xi|^{-b}$ where $\chi_{\infty}$ is smooth, equal to 1 for large $\xi$ and $\chi_{\infty}(0) = 0$; moreover $q$ is a homogeneous function of degree $a > 0$, smooth away from the origin, and $b > qa/2$.

   d*) same multiplier as in (c), with the assumption of $a = 1$ (for example $q(\xi) = |\xi|$). Then $m \in M_1$ under the weaker condition $b > (n-1)/2$.

   (This is harder than (c). For the easier case $n = 1$ compare (b)).

e) $m(\xi) = (1 - |\xi|^2)^{\frac{a}{2}}$ for $\lambda > (n-1)/2$.

   f) $m(\xi) = (1 - \rho(\xi))\frac{x}{2}$ for $\lambda > (n-1)/2$, where $\rho$ is sufficiently smooth (say, $\rho$ is $N$ times differentiable with $N > [n/2]$).

10. Suppose that $m \in C^{N+1}([0, \infty))$, $s^km^{(k)}(s) \to 0$ if $s \to \infty$, $1 \leq k \leq N$, and

   \[(*) \quad \int_0^\infty s^N |m^{(N+1)}(s)|ds < \infty.\]

   (i) Show that $m(t) = \frac{1}{\Gamma(N)} \int_t^\infty (s-t)^N m^{(N+1)}(s)ds$ for $t > 0$.

   (ii) Use problem 9e or 9f above to show that condition (*) for $N > (n-1)/2$ implies that the radial function $m(|\cdot|)$ is the Fourier transform of an $L^1$ function.

   (iii) Let $a > 0$ and

   \[
   m_1(\xi) = \frac{|\xi|^a}{(1 + |\xi|^2)^{a/2}}, \quad m_2(\xi) = \frac{(1 + |\xi|^2)^a}{(1 + |\xi|^2)^{a/2}}, \quad m_3(\xi) = \frac{(1 + |\xi|^2)^a/2}{(1 + |\xi|^2)^{a/2}}.
   \]

   Show that $m_1$, $m_2$ and $m_3$ are Fourier transforms of bounded Borel measures. (Hint: The calculation must involve subtracting 1).

11. Let $1 \leq q \leq 2$.

   (i) Let $N > n(1/q - 1/2)$. Suppose that $m$ has derivatives up to order $N$ which lie in $L^2$. Then $m \in L^q$. 

   (ii) The hypothesis in (i) can be replaced by $m \in L^2_{\alpha}$ where $\alpha > n(1/q - 1/2)$ and $L^2_{\alpha}$ is the $L^2$-Sobolev space.

   (iii) Find dilation invariant criteria similar to the one in problem 10.

   (A dilation invariant criterion for membership in $M^{p,q}$ should reflect the formula $t^{n(1/p-1/q)}\|m(t)\|_{M^{p,q}} = \|m\|_{M^{p,q}}$).

12. Suppose $1 \leq p \leq r'$ and suppose that $K \in \text{Weak-}L^r$. Suppose that $1/p - 1/q = 1 - 1/r'(r > 0)$. Show that the convolution operator $f \mapsto K*f$ is of weak type $(p,q)$. If in addition $1 < p < q < \infty$ then the operator is of strong type $(p,q)$.

13. (i) Formulate and prove versions of the Marcinkiewicz interpolation theorem involving the space $L^\infty$ for one of the endpoints.

   (ii) What can you say about the behavior of the constants if in the hypothesis of the Marcinkiewicz interpolation theorem if one of the weak type estimates is replaced by a strong type estimate?
14. Show that one can define a tempered distribution $u$ by

$$u = p.v. \frac{1}{x}$$

i.e.

$$\langle u, \phi \rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{\phi(x)}{x} dx$$

and compute the Fourier transform of $u$. 