Faber expansions and sampling in mixed order Sobolev spaces

Glenn Byrenheid

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We consider the mixed order Sobolev space
\[ S_{r}^{p}W(T^{d}) := \left\{ f \in L_{p}(T^{d}) : \left\| \left( \sum_{j \in \mathbb{N}^{d}_{0}} 2^{2r|j|+1} |\delta_{j}[f](\cdot)|^{2} \right)^{1/2} \right\|_{p} < \infty \right\}, \]
where \( 1 < p < \infty \) and \( \frac{1}{p} < r < \infty \). Here \( \delta_{j}[f] \) is that part of the Fourier series of \( f \) with frequencies in a dyadic anisotropic rectangle. We study a replacement of \( \delta_{j}[f] \) by building blocks that use only discrete information of \( f \) (function evaluations). Such a replacement (in the sense of equivalent norms) can be achieved with the help of tensorized Faber basis where a continuous function \( f \) is decomposed into tensor products of dilated and translated hat functions. Here we need the condition \( r > \frac{1}{p} \). The obtained discrete characterization is well suited for studying sampling issues in \( S_{r}^{p}W(T^{d}) \). We construct a sampling algorithm taking values on a sparse grid that allows for proving asymptotically optimal error bounds for the linear sampling numbers
\[ g_{n}(S_{r}^{p}W(T^{d}), Y) = \inf_{\{\xi_{i}\}_{i=1}^{n} \subset T^{d}} \sup_{\{\psi_{i}\}_{i=1}^{n} \subset Y} \| f(\cdot) - \sum_{i=1}^{n} f(\xi_{i}) \psi_{i}(\cdot) \|_{Y}, \]
where the error is measured in a Lebesgue space \( L_{q}(T^{d}) \), \( 1 < q \leq \infty \) as well as in isotropic Sobolev spaces \( W_{q}^{s}(T^{d}) \), \( s > r \), \( 1 < p \leq q < \infty \). The talk is based on a joint work with Tino Ullrich.