Construct a bounded set in $\mathbb{R}$ with exactly three limit points.

I wish to prove that the set $S = \{x \in \mathbb{R} : 1/n, n \in \mathbb{N}\} \cup \{x \in \mathbb{R} : 1/n + 1, n \in \mathbb{N}\} \cup \{x \in \mathbb{R} : 1/n + 2, n \in \mathbb{N}\}$ is bounded and has exactly three limit points.

Consider the neighborhood of radius 3 around the point 1, denoted $N_3(1)$. Since $1/n > 0, \forall n \in \mathbb{N}$, then $s > 0, \forall s \in S$. Also, since $1/n \leq 1, \forall n \in \mathbb{N}$, then $s \leq 2, \forall s \in S$. Therefore, $S \subset N_3(1)$, meaning $S$ is bounded.

Next, consider the point 0. For any radius $r$, we can find a point in $S$ by taking $n$ such that $1/n < r$. Next consider the point 1. For any radius $r$, we can find a point in $S$ by taking $n$ such that $1/n + 1 < r + 1$. Similarly, around the point 2, for any radius $r$ we can find a point in $S$ by taking $n$ such that $1/n + 2 < r + 2$. Therefore the points 0, 1, and 2 are limit points.

Now, take $s \in S - \{0, 1, 2\}$. Then, $s$ is of the form $s = 1/n + z, n \in \mathbb{N}, z \in \{0, 1, 2\}$. This means that for any $s$, there are no other points in $S$ in the neighborhood $N_{1/(n+1)}(s)$. Therefore, $s$ is not a limit point.

Thus, the set $S$ is bounded with exactly 3 limit points.

2.11 Metrics

For $x, y \in \mathbb{R}$, determine if the following functions are metrics.

(1) $d_1(x, y) = (x - y)^2$.

Checking the triangle inequality property, pick some arbitrary $z \in \mathbb{R}$. Then,

$d_1(x, z) = (x - z)^2 = x^2 - 2xz + z^2$

and

$d_1(x, y) + d_1(y, z) = (x - y)^2 + (y - z)^2$

$= x^2 - 2xy + y^2 + y^2 - 2yz + z^2$

$= x^2 - 2xy + 2y^2 - 2yz + z^2$

$(d_1(x, y) + d_1(y, z)) - d_1(x, z) = x^2 - 2xy + 2y^2 - 2yz + z^2 - x^2 + 2xz - z^2$

$= 2xz - 2xy - 2yz + 2y^2$

Thus, picking $z = 0$ and $0 < y < x$ breaks the triangle inequality property. Therefore, $d_1$ is not a metric.

(2) $d_2(x, y) = \sqrt{|x - y|}$

$\sqrt{z} = 0$ if and only if $z = 0$, and $|x - y| = 0$ if and only if $x = y$. Thus, $d_2(x, y) = 0$ if and only if $x = y$. Also, for $z > 0$, $\sqrt{z} > 0$. Thus, $d_2(x, y) > 0$ if $x \neq y$.

Since $|x - y| = |y - x|$, then $\sqrt{|x - y|} = \sqrt{|y - x|}$.

Finally, we will show that the triangle inequality holds for $d_2$. Let $z \in \mathbb{R}$. Note that the first statement is true by Thm 1.37 in Rudin.

$|x - y| + |y - z| \geq |x - z|$

$|x - y| + 2\sqrt{|x - y||y - z|} + |y - z| \geq |x - z|$

$(\sqrt{|x - y|} + \sqrt{|y - z|})^2 \geq |x - z|$

$\sqrt{|x - y|} + \sqrt{|y - z|} \geq \sqrt{|x - z|}$

$d_2(x, y) + d_2(y, z) \geq d_2(x, z)$

Therefore, all the properties of a metric are held, and $d_2$ is a metric.

(3) $d_3(x, y) = |x^2 - y^2|$
No.
For \( x = 1, y = -1 \)
\[
d_3(1, -1) = |x^2 - y^2| = |1^2 - (-1)^2| = 0
\]
Which violates the property \( d(x, y) \neq 0 \) for \( x \neq y \).

\((d_4)\)

No.
For \( x = 1 \)
\[
d_4(x, x) = |x - 2x| = 1
\]
Which violates the property \( d(x, x) = 0 \).

\((d_5)\)

Yes.
(1)
\[
d_5(x, x) = \frac{|x - x|}{1 + |x - x|} = \frac{0}{1 + 0} = 0.
\]
(2) By \( |x - y| = |y - x| \)
\[
d_5(x, y) = \frac{|x - y|}{1 + |x - y|} = \frac{|y - x|}{1 + |y - x|} = d_5(y, x).
\]
(3)
\[
\frac{p}{1 + p} \leq \frac{q}{1 + q} + \frac{r}{1 + r}
\]
\[\Leftrightarrow \ p(1 + q)(1 + r) \leq q(1 + r)(1 + p) + r(1 + p)(1 + q)\]
\[\Leftrightarrow \ p + pq + pr + pqr \leq (q + pq + qr + pqr) + (r + pr + qr + pqr)\]
\[\Leftrightarrow \ p \leq q + r + 2qr + pqr.
\]
Because \( |x - z| \leq |x - y| + |y - z| \), it follows that \( d_5(x, y) \) satisfies (3) and is, therefore, a metric.