1. Vector Spaces

Show that the following sets are vector spaces over the following fields.

(a) Show that the set of complex numbers, $\mathbb{C}$, is a vector space over the field of real numbers $\mathbb{R}$ where addition is simply addition of complex numbers and scalar multiplication is:

$$\alpha \cdot (a + bi) = (\alpha \cdot a + \alpha \cdot bi)$$

for $\alpha, a, b \in \mathbb{R}$.

(b) Let $V = \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of all functions from the real numbers to the real numbers. Show that $V$ is a vector space over $\mathbb{R}$ where addition and multiplication are defined below:

$$(f + g)(x) = f(x) + g(x),$$

$$(\alpha \cdot f)(x) = \alpha \cdot f(x),$$

for all $f, g \in V, x, \alpha \in \mathbb{R}$.

(c) Let $V = \{f : \mathbb{F}_p \to \mathbb{C}\}$ be the set of all functions from the finite field with $p$ elements to the complex numbers. Show that $V$ is a vector space over $\mathbb{C}$, where addition and multiplication are defined below:

$$(f + g)(x) = f(x) + g(x),$$

$$(\alpha \cdot f)(x) = \alpha \cdot f(x),$$

for all $f, g \in V, \alpha \in \mathbb{C}$, and $x \in \mathbb{F}_p$.

2. Subspaces

Let $V$ be a vector space over $F$. Let $W \subset V$ be a subset of the vectors in $V$. We call $W$ a subspace of $V$, if $W$ is also a vector space using the SAME addition and multiplication rules given in $V$, and $0 \in W$. (Here 0 is the zero vector of $V$).

Fact: If $W \subset V$, in order to prove that $W$ is a subspace of $V$, one must only show the following three things:
(a) For all \( u, v \in W, u + v \in W \). (Closure of Addition)
(b) For all \( u \in W \) and \( \alpha \in F, \alpha \cdot u \in W \). (Closure of Scalar Multiplication)
(c) \( (0) \in W \). (Where 0 is the zero vector of \( V \)).

We will now prove several facts about subspaces.

(a) Let \( U, W \) be subspaces of a vector space \( V \) over a field \( F \). The intersection of \( U \) and \( W \) is defined as \( U \cap W = \{ v \in V \mid v \in U, v \in W \} \). Prove that \( U \cap W \) is a subspace of \( V \).

(b) Let \( V \) be a vector space over a field \( F \). Let \( S \subset V \) be a subset (not necessarily subspace) of \( V \). The span of \( S \) is defined as:

\[
\text{Span}(S) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \mid \alpha_j \in F, v_i \in S, \text{ for every } j \}.
\]

We will show in class that the span of a set is a subspace. Now let \( V = \mathbb{R}^3 \). Let \( U = \{(x_1, x_2, x_3) \in V \mid x_1 + x_2 + x_3 = 0\} \), \( W = \{(x_1, x_2, x_3) \in V \mid x_2 = 0\} \), and \( v = (1, 0, -1) \in V \). Show that:

\[ U \cap W = \text{Span}\{v\} \]

(c) For the following vector spaces \( V \) and subsets \( W \subset V \) determine whether or not \( W \) is a subspace of \( V \). (Simply answer "yes" or "no", no proof is needed)

1. \( V = \mathbb{R}^3 \). \( W = \{(x_1, x_2, x_3) \in V \mid x_3 = 0\} \)
2. \( V = \mathbb{R}^3 \). \( W = \{(x_1, x_2, x_3) \in V \mid x_1, x_2, x_3 \in \mathbb{Z}\} \) (Here, \( \mathbb{Z} \) denotes the set of integers)
3. \( V = \{f : \mathbb{R} \rightarrow \mathbb{R}\}. \ W = \{f \in V \mid f(0) = 0\} \)
4. \( V = \{f : \mathbb{R} \rightarrow \mathbb{R}\}. \ W = \{f \in V \mid f(0) = 1\} \)
5. Let \( F \) be a field. Let \( V = M_2(F) \) be the vector space of two-by-two matrices with entries in \( F \). Let

\[ W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \mid a + d = 0 \right\}. \]

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!