Remark. Answers should be written in the following format:
A) Result.
B) If possible the name of the method you used.
C) The computation.

1. **Bases & Dimension**

Let $V$ be a vector space over a field $F$. If there exists a finite set of vectors, $S = \{v_1,v_2,...,v_n\}$ that spans $V$, then we say that $V$ is **finitely generated**.

Suppose $B = \{v_1,v_2,...,v_n\} \subset V$ is a linearly independent subset that generates $V$. Then we call $B$ a **basis** for $V$. It will be shown in class that any two bases for $V$ have the same size. It will be shown in class that in this case the **dimension** of $V$ is the number of elements $|B| = n$.

Let $B \subset V$, an $n$-dimensional vector space. The following conditions are equivalent

(a) $B$ is a basis for $V$.
(b) $B$ is linearly independent, and $|B| = n$.
(c) $B$ generates $V$, and $|B| = n$.
(d) $B$ is a minimal spanning set (namely, $B$ spans $V$ and no proper subset $S \subset B$ spans $V$).
(e) $B$ is a maximal linearly independent set (namely, $B$ is linearly independent and no $S$ which properly contains $B$ is linearly independent).

The following questions will help you become familiar with this equivalence.

(a) The complex numbers $\mathbb{C}$ form a 2-dimensional vector space over $\mathbb{R}$. Let $B = \{1, i\}$ in $\mathbb{C}$. Show that $B$ is a basis for $\mathbb{C}$ by showing that it generates $V$ and using part (c) of the theorem above.

(b) Let $F$ be any field. Let $F^n = \{(x_1,...,x_n)\mid x_j \in F \text{ for every } j\}$ be the vector space over $F$ with natural coordinate wise operations. Find a basis for $F^n$ by constructing a generating set, and reducing it until it’s linearly independent. Conclude that $F^n$ is $n$ dimensional.

(c) Let $V = \mathbb{R}_{\leq 3}[x] = \{a_3x^3 + a_2x^2 + a_1x + a_0\mid a_i \in \mathbb{R}, \forall i\}$, the set of polynomials of degree at most 3, with real coefficients. This vector space is 4-dimensional. Consider the set $B = \{1, x + 1, x^2 + x + 1, x^3 + x^2 + x + 1\}$. Show that this set is linearly independent and conclude that it is a basis using part (b) of the theorem.
(d) Let $V = \mathbb{R}^3$. Show that the set,

$$B = \{ \begin{pmatrix} 1 & 3 & 4 \\ -2 & 1 & 1 \\ 0 & 7 & 8 \end{pmatrix} \}$$

is a basis for $V$, by demonstrating any one of the five criteria above.

(e) Let $V = M_2(\mathbb{C})$, the set of two by two matrices over the field $\mathbb{C}$. The set,

$$B = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \}$$

generates $V$. Show that $B$ is a basis for $V$ by showing that any proper subset of $B$ does not generate $V$.

2. Coordinates

Let $V$ be a vector space over $F$ and let $B = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis. Let $u \in V$. Since $B$ is a basis there is exactly one solution $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in F^n$ to the equation,

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n = u.$$ 

We call this vector,

$$[u]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

the coordinates of $u$ with respect to $B$.

Let $V = \mathbb{R}^4$, $W = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 + x_2 + x_3 - x_4 = 0 \}$, and let

$$B = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}.$$

(a) Prove that $B$ is a basis for $W$.

(b) Let

$$u = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 4 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}.$$

Find $[u]_B$, $[v]_B$, and $[u+v]_B$. Add the first two, and notice that $[u]_B + [v]_B = [u+v]_B$.

(c) We would like to generalize this phenomenon. So now consider any $w, w' \in W$. Show that $[w]_B + [w']_B = [w+w']_B$. 

2
Remarks

- You are very much encouraged to work with other students. However, submit your work alone.

- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!