1. The Matrix of a Linear Transformation

Let $T : V \to W$ be a linear transformation between vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for $V$ and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be a basis for $W$. Then we have a theorem that there is a unique matrix $[T]_{\mathcal{C}, \mathcal{B}} \in M_{m \times n}(\mathbb{F})$ such that, for all $v \in V$,

$$[T(v)]_\mathcal{C} = [T]_{\mathcal{C}, \mathcal{B}} \cdot [v]_\mathcal{B}.$$ 

Moreover, the matrix $[T]_{\mathcal{C}, \mathcal{B}}$ is given by the equation,

$$[T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(v_1)]_\mathcal{C} & \cdots & [T(v_n)]_\mathcal{C} \end{bmatrix}.$$ 

In this problem you will compute several such matrices and prove a basic property.

(a) Let $V = \mathbb{R}_{\leq 2}$, with the basis $\mathcal{B} = \{1, x, x^2\}$. Let $T : V \to V$ be given by the equation,

$$T(p(x)) = x \cdot p'(x)$$

Compute $[T]_{\mathcal{B}, \mathcal{B}}$ (Notation: when we have the same basis for the domain and codomain we just write $[T]_\mathcal{B}$ rather than $[T]_{\mathcal{B}, \mathcal{B}}$).

(b) Let $V$ be a vector space over a field $\mathbb{F}$, with the basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$. Let $\alpha \in \mathbb{F}$. Let $T : V \to V$, where $T(v) = \alpha \cdot v$. Compute $[T]_\mathcal{B}$.

(c) Prove the uniqueness part of the above theorem. In other words, suppose you have two matrices $A, B \in M_{m \times n}(\mathbb{F})$ such that, for all $v \in V$,

$$A \cdot [v]_\mathcal{B} = B \cdot [v]_\mathcal{B}.$$ 

Show that $A = B$. In particular, explain why this implies the uniqueness part.
2. Changing Bases

In this problem our final goal is to prove a relationship between the matrices of linear transformations with respect to different bases. Let $V$ be a vector space and let $\mathcal{B}$ and $\mathcal{C}$ be bases for $V$. Let $T : V \to V$ be a linear transformation. We will prove in this problem that,

$$[T]_\mathcal{B} = (M_{\mathcal{C},\mathcal{B}})^{-1} \cdot [T]_\mathcal{C} \cdot M_{\mathcal{C},\mathcal{B}}$$

We will do this in several steps. I suggest working with the definitions of these matrices and not doing explicit computations.

(a) Show that $[T]_{\mathcal{C},\mathcal{B}} = [T]_{\mathcal{C}} \cdot M_{\mathcal{C},\mathcal{B}}$.
(b) Show that $[T]_\mathcal{B} = M_{\mathcal{B},\mathcal{C}} \cdot [T]_{\mathcal{C},\mathcal{B}}$. So by these two equations, we have

$$[T]_\mathcal{B} = (M_{\mathcal{B},\mathcal{C}}) \cdot [T]_{\mathcal{C}} \cdot M_{\mathcal{C},\mathcal{B}}.$$

(c) Lastly, show that $M_{\mathcal{B},\mathcal{C}} = (M_{\mathcal{C},\mathcal{B}})^{-1}$, by showing that

$$M_{\mathcal{B},\mathcal{C}} \cdot M_{\mathcal{C},\mathcal{B}} = I = M_{\mathcal{C},\mathcal{B}} \cdot M_{\mathcal{B},\mathcal{C}}.$$

3. Compositions of Linear Transformations

Let $V, W,$ and $U$ be vector spaces over the field $\mathbb{F}$. Let $T : V \to W$ and $S : W \to U$ be linear transformations. Prove the following facts about the linear transformation $S \circ T : V \to U$.

(a) $\dim(\ker(S \circ T)) \geq \dim(\ker(T))$.
(b) $\dim(\im(S \circ T)) \leq \min\{\dim(\im(T)), \dim(\im(S))\}$.
(c) Let $V = W = U = \mathbb{R}^2$. Find linear transformations $T, S$ such that neither $T$ nor $S$ is the zero-transformation, but $S \circ T$ is the zero transformation.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!