1. \[ T : V \to V \quad \text{and} \quad B = \{ e_1, x_1, x_2 \} \]

\[
[T]_B = \begin{pmatrix} [T(x_1)]_B & [T(x_2)]_B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

2. \[ T : V \to V \quad B = \{ v_1, \ldots, v_n \} \]

\[
[T]_B = \begin{pmatrix} [T(v_1)]_B & [T(v_2)]_B \ldots & [T(v_n)]_B \end{pmatrix} = \begin{pmatrix} \alpha v_1 & \alpha v_2 & \ldots & \alpha v_n \end{pmatrix} = \begin{pmatrix} \alpha \alpha \ldots & 0 \\ 0 & \alpha \ldots & 0 \\ \vdots & \ddots & \alpha \end{pmatrix}
\]

3. Suppose \( A \in \text{Mat}_{n \times n}(F) \) and \( \forall \ v \in V \) (n-dimensional vs, over F) \( A \cdot [v]_B = B \cdot [v]_B \). Then \( A = B \).

**Proof**

Let \( e_i \in F^n \) be the \( i \)th standard basis vector. Denote the basis vectors in \( B = \{ v_1, \ldots, v_n \} \); \( e_i = [v_i]_B \). So

\[ A e_i = A [v_i]_B = B [v_i]_B = B e_i \]

But the left hand side is the \( i \)th column of \( A \) and the right hand side is the \( i \)th column of \( B \). So the entries of \( A \) are the same as the entries of \( B \), so \( A = B \).
This lemma is applicable to the given theorem because it acts like $[T]_{eB}$, it must be $[T]_{eB}$, thus proving uniqueness.

2. Claim: $[T]_{eB} = [T]_e \cdot M_{eB}$

Proof:
\\forall v \in V, ([T]_e \cdot M_{eB})[v]_B = [T]_e[v]_e = [T(v)]_e = [T]_e B[v]_B. So by 1c, $[T]_e \cdot M_{eB} = [T]_{eB}$.

3. Claim: $[T]_B = M_{Be} \cdot [T]_{eB}$

Proof:
\\forall v \in V, M_{Be} \cdot [T]_{eB} \cdot [v]_B = M_{Be} \cdot [T(v)]_e = [T(v)]_B = [T]_B \cdot [v]_B. So by 1c, $M_{Be} \cdot [T]_{eB} = [T]_B$.

So far we have $[T]_B = M_{Be} \cdot [T]_e \cdot M_{eB}$

4. Claim: $M_{Be} = (M_{eB})^{-1}$

Proof:
Let $v \in V$, $M_{Be} \cdot M_{eB} \cdot [v]_B = M_{Be} \cdot [v]_e = [v]_B = I \cdot [v]_B$. So $M_{Be} \cdot M_{eB} = I$. By an identical argument we have $M_{eB} \cdot M_{Be} = I$. So $M_{Be} = (M_{eB})^{-1}$.

So we have,

$[T]_B = (M_{eB})^{-1} \cdot [T]_e \cdot M_{eB}$
Claim: $\dim(\ker(S \circ T)) \geq \dim(\ker(T))$

Proof: Let $v \in \ker(T)$. Then $S \circ T(v) = S(T(v)) = S(0v) = 0v$. So $v \in \ker(S \circ T)$. So $\ker(T) \subseteq \ker(S \circ T)$. So $\dim(\ker(T)) \leq \dim(\ker(S \circ T))$.

Claim: $\dim(\text{im}(S \circ T)) \leq \min\{\dim(\text{im}(S)), \dim(\text{im}(T))\}$

Proof: First we'll show $\dim(\text{im}(S \circ T)) \leq \dim(\text{im}(S))$. Let $u \in \text{im}(S \circ T)$. Then $\exists v \in V$ such that $u = S \circ T(v) = S(T(v))$ with $T(v) \in W$ so $u \in \text{im}(S)$. So $\text{im}(S) \supseteq \text{im}(S \circ T)$. So $\dim(\text{im}(S)) \geq \dim(\text{im}(S \circ T))$.

Now we'll show $\dim(\text{im}(S \circ T)) \leq \dim(\text{im}(T))$. By Rank-Nullity $\dim(\text{im}(S \circ T)) = \dim(\text{im}(T)) - \dim(\ker(S \circ T)) = \dim(\text{im}(T)) - \dim(\ker(T)) = \dim(\text{im}(T))$ using 3a above.

So $\dim(\text{im}(S \circ T)) \leq \min\{\dim(\text{im}(S)), \dim(\text{im}(T))\}$

$\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{S} \mathbb{R}^2$ we want $T, S \neq 0$ but $S \circ T = 0$.

Take $T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $S(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} y \\ 0 \end{pmatrix}$.

Then $S \circ T(\begin{pmatrix} x \\ y \end{pmatrix}) = S(T(\begin{pmatrix} x \\ y \end{pmatrix})) = S(\begin{pmatrix} x \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

\[
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
V & \xrightarrow{T} & W & \xrightarrow{S} \mathbb{R}^2 \\
(\xi) & \mapsto & (\xi) & \mapsto (\xi)
\end{array}
\]