1. **The Matrix of a Linear Transformation.** Let $B = \{v_1, \ldots, v_n\}$ be a basis for a vector space $V$ over $F$. The coordinate map $[\cdot]_B : V \to F^n$ is the map that assigns to a vector $v \in V$ the vector

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

where $\alpha_i \in F$ are the unique scalars such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

(a) Let $T : V \to V$ be a linear transformation between vector spaces over a field $F$. Show that there is a unique matrix $[T]_B \in M_n(F)$ such that for all $v \in V$

$$[T(v)]_B = [T]_B \cdot [v]_B.$$

Moreover, show that the matrix $[T]_B$ has the formula

$$[T]_B = \begin{pmatrix} [T(v_1)]_B & \cdots & [T(v_n)]_B \end{pmatrix}.$$

(b) Let $V = \mathbb{R}_{\leq 2}[x]$ the vector space of polynomials of degree at most two with coefficients in $\mathbb{R}$. Let $T : V \to V$ be given by the equation

$$T(p(x)) = (xp(x))'.$$

Compute $[T]_B$ and $[T]_C$ where $B = \{1, x, x^2\}$ and $C = \{1 - x, 1 + x, 1 + x^2\}$.

2. **Eigenvalues and eigenvectors.** Let $T : V \to V$ be a linear transformation. Show that if $v_1, \ldots, v_k \in V$ are eigenvectors of $T$ associated with different eigenvalues $\lambda_1, \ldots, \lambda_k$, then they are linearly independent.

3. **Direct Sum.** We say that a vector space $V$ is a direct sum of the subspaces $V_1, \ldots, V_k \subset V$, and denote $V = V_1 \oplus \cdots \oplus V_k$, if every $v \in V$ can be written uniquely as $v = v_1 + \cdots + v_k$, where $v_i \in V_i$ for every $i = 1, \ldots, k$.

(a) Suppose $V$ is finite dimensional vector space and $V_1, \ldots, V_k \subset V$ subspaces. Show that the following are equivalent:
1. $V = V_1 \oplus \ldots \oplus V_k$.

2. For every collection of basis $B_1$ for $V_1, \ldots, B_k$ for $V_k$, their union $B = B_1 \cup \ldots \cup B_k$ is a basis for $V$.

(b) Suppose $V$ has dimension $n$. Let $T : V \to V$ be a linear transformation. Show that the following are equivalent:

1. There exists a basis $B = \{v_1, \ldots, v_n\}$ for $V$ such that
   \[
   [T]_B = \begin{pmatrix}
   \lambda_1 \\
   \vdots \\
   \lambda_n
   \end{pmatrix}.
   \]

2. There exists $\mu_1, \ldots, \mu_k \in \mathbb{F}$ and subspaces $V_1, \ldots, V_k$, such that $V = V_1 \oplus \ldots \oplus V_k$, and for each $i = 1, \ldots, k$ the action of $T$ on $V_i$ is given by multiplication by $\mu_i$.

**Rings.**

4. Let $\mathbb{F}$ be a field and $\mathbb{F}[X]$ the ring of polynomials with coefficients in $\mathbb{F}$.

   (a) Show that $\dim \mathbb{F}[X] = \infty$.

   (b) Recall that the degree $\deg(f)$ of a polynomial $f \in \mathbb{F}[X]$ is defined to be $\deg(f) = d$ if $f = a_d X^d + \ldots + a_1 X + a_0$ and $a_d \neq 0$, and $\deg(f) = -\infty$ if $f = 0$. Show that $\deg(fg) = \deg(f) + \deg(g)$ for every $f, g \in \mathbb{F}[X]$.

5. Let $\varphi : R \to S$ be homomorphism of rings. Define the kernel of $\varphi$, denoted $\ker(\varphi)$, by $\ker(\varphi) = \{a \in R \text{ such that } \varphi(a) = 0\}$. Show that $\varphi$ is one-to-one if and only if $\ker(\varphi) = \{0\}$.

6. Show that if $R$ is a ring with unit and $a \in R$ is invertible, then it has a unique inverse.

**Remark**

The grader and the Lecturer will be happy to help you with the homework. Please visit the office hours.

**Good Luck!**