1. Let $T : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ be the linear transformation defined by
\[
(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_3, -x_1 - x_3, x_2 + x_4, x_2 - x_4).
\]
Find all eigenvalues of $T$ and for each eigenvalue, compute the corresponding eigenspace.

2. For each of the possible values of the matrix $A \in M_3(\mathbb{Q})$ below,
\[
\begin{pmatrix}
3 & 1 & 1 \\
2 & 4 & 2 \\
1 & 1 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & -1 \\
-1 & 1 & 4
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
find their eigenvalues (in $\mathbb{Q}$) and compute the corresponding eigenspaces. Then decide if there exists an invertible $3 \times 3$ matrix $C$ such that $C^{-1}AC$ is diagonal. If such a matrix $C$ exists, find $C$ and compute $C^{-1}AC$.

3. (a) Let $\mathbb{F}$ be an algebraically closed field. Let $V$ be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear transformation. Show that $T$ has at least one eigenvalue.

(b) Find an example of a non-algebraically closed field $\mathbb{F}$, a vector space $V$ over $\mathbb{F}$, and a linear transformation $T : V \to V$ such that $T$ has no eigenvalues.

4. Let $P : V \to V$ be a linear transformation on a vector space $V$ over a field $\mathbb{F}$ such that $P^2 = P$. Show that $P$ is diagonalizable.
5. Let $T : V \to V$ be a linear transformation, where $V$ is a vector space of dimension $n$ over a field $\mathbb{F}$. A subspace $W \subset V$ is called $T$-invariant if $T(W) \subset W$. Suppose that in addition we have another operator $S : V \to V$ and that $S$ and $T$ commute, i.e. $ST = TS$. Show that

(a) If $W = W_\mu$ is an eigenspace for $S$ associated with the eigenvalue $\mu \in \mathbb{F}$, then $W$ is $T$-invariant.

(b) If $S, T$ are each diagonalizable, then they are simultaneously diagonalizable, i.e., there exist scalars $\mu_i, \lambda_i \in \mathbb{F}$, for $i = 1, \ldots, k$, and a direct sum decomposition

$$V = \bigoplus_{i=1}^{k} V_i,$$

such that $S|_{V_i} = \mu_i Id_{V_i}$ and $T|_{V_i} = \lambda_i Id_{V_i}$ for every $i = 1, \ldots, k$.

Remark
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!