Math 341 - Fall 2014
Preparation for Final Test

Remarks

- Answer All the questions below.
- A definition is just a definition – there is no need to justify it. Just write it down.
- Unless it’s a definition, answers should be written in the following format:
  - Write the main points that will appear in your explanation or proof or computation. Main points:........
  - Write the actual explanation or proof or computation. Proof:........ or Computation:........

1. Matrix Presentation of Linear Transformation.

(a) (8) Let $V$ and $W$ be vector spaces of dimension $n$ and $m$, respectively, over a field $F$. For a linear transformation $T : V \rightarrow W$ and ordered bases $B = \{v_1, \ldots, v_n\}$ for $V$, and $C = \{w_1, \ldots, w_m\}$ define the matrix $[T]_{C,B}$ representing $T$ with respect to $B$ and $C$, and write a formula for it.

(b) (15) Use the definition above to show that for $B$ and $C$ basis for $V$, and $T : V \rightarrow V$, we have the relation

$$[T]_C = M_{C,B} \cdot [T]_B \cdot M_{B,C},$$

where $[T]_B$ is the notation for $[T]_{B,B}$ (Hint: It is enough to show that the left and right hand sides agree on a general vector of the form $[v]_C \in F^n$. You can use the fact that the change of basis matrix $M_{B,C} \in M_n(F)$ is defined by the property $M_{B,C} \cdot [v]_C = [v]_B$, for every $v \in V$).

(c) (10) Let $V$ be a vector of dim($V$) = 5. Let $B = \{v_1, \ldots, v_5\}$ be a basis for $V$. Consider the linear transformation $T : V \rightarrow V$, given by

$$Tv_1 = 0,\;Tv_2 = v_1,\;\ldots,\;Tv_5 = v_4.$$ 

Compute the matrix $A = [T]_B$, and show that $A^5 = 0$ without using matrix multiplication (Hint: recall the multiplicativity property of the map $[\cdot]_B : L(V) \rightarrow M_5(F)$).

2. Invertibility.

(a) (8) Define when a linear transformation $T : V \rightarrow W$ is invertible. Define when a matrix $A \in M_n(F)$ is invertible.

(b) (15) Let $V$ be a vector space of dimension $n$ over a field $F$. Let $B$ be a basis for $V$. Show that a linear transformation $T : V \rightarrow V$ is invertible if and only if the matrix $[T]_B$ is invertible.
(c) (10) Let
\[ A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R}). \]

Show that \( A \) is invertible and compute its inverse. Deduce that the linear transformation \( T_A : \mathbb{R}^3 \to \mathbb{R}^3, v \mapsto A \cdot v \), is invertible. What is its inverse?

3. **Determinant.**

   (a) (8) Formulate the multiplicativity property of the determinant of a linear transformation.

   (b) (15) Show that if \( T : V \to V \) is invertible linear transformation then \( \det(T^{-1}) = \det(T)^{-1} \).

   (c) (10) Compute the determinant of the linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \) given by \( Te_1 = e_n, Te_2 = e_{n-1}, \ldots, Te_n = e_1 \), where \( \mathcal{B}_{st} = \{ e_j ; 1 \leq j \leq n \} \) is the standard basis of \( \mathbb{R}^n \) (Hint: \( 1 + \ldots + n = \frac{n(n+1)}{2} \)).

4. **Diagonalization.**

   (a) (8) For a linear transformation \( T : V \to V \), where \( \dim(V) = n \), define the notions of eigenvalue, eigenvector, and what does it mean that \( T \) is diagonalizable.

   (b) (15) Prove that if the above \( T \) has \( n \) distinct eigenvalues then it is diagonalizable (Hint: You may use the fact that eigenvectors with different eigenvalues are linearly independent).

   (c) (8) Let
\[ A = \begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}. \]

What is \( A^{2014} =? \).

**Good Luck!!**