There are 7 problems, and a total of 10 pages. If you use the extra pages at the end, clearly indicate that you have done so, and clearly label the extra pages with the corresponding problem number.

**Instructions:** Solve each problem, completely and carefully justifying each deduction. You may **not** consult any books, notes, peers, internets, or other outside resources. Cheating will not be tolerated.

**Grading notes:** Problems will be graded on correctness and completeness of the solution, but also on style (so write in complete sentences). Partial credit will be given for correct definitions of the relevant terms, statements of relevant theorems, helpful pictures, or other positive progress. If you get stuck, tell me what you do know (within reason). Credit is **not** based on the length of your answer, so write proofs, not essays.
1. (10 pts) Let $E$ be a nonempty subset of $\mathbb{R}$ and let $\lambda > 0$. Define

$$\lambda E = \{ \lambda x : x \in E \}.$$ 

Prove that $\sup(\lambda E) = \lambda \sup E$. 

2. (10 pts) Determine whether the series

\[ \sum_{n=1}^{\infty} \frac{n2^n}{4^n + (-1)^n} \]

converges. Prove your answer.
3. (10 pts) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Prove directly from the definitions that if \(f : X \to Y\) is continuous and \(G \subseteq Y\) is open, then \(f^{-1}(G)\) is open.

Then, given an example of a continuous function \(f : \mathbb{R} \to \mathbb{R}\) and an open set \(G\) such that \(f(G)\) is not open.
4. (10 pts) Use the definition of uniform continuity to prove that $f(x) = x^3$ is uniformly continuous on $[0, 1]$ but not on $[0, \infty)$. 

Hint: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. 
5. (10 pts) Let $a < b$, and let $f$ be a continuous function on $[a, b]$. Prove that there exists a point $c \in (a, b)$ such that

$$\int_{a}^{b} f \, dx = f(c)(b - a).$$

Give the precise statement of any theorem(s) that you use.
6. (10 pts) Let $E$ be a set and $(f_n)$ and $(g_n)$ be functions, $f_n : E \to \mathbb{R}$ and $g_n : E \to \mathbb{R}$. Assume that $f_n \to f$ uniformly on $E$ and $g_n \to g$ uniformly on $E$. Prove that $f_n + g_n \to f + g$ uniformly on $E$. 
7. (14 pts) a. Give an example of a sequence \((a_n)\) with infinite range (i.e. \(\{a_n : n \in \mathbb{N}\}\) is not a finite set) for which the limit superior and limit inferior (\(\lim\sup\) and \(\lim\inf\)) are different. Calculate \(\lim\sup a_n\) and \(\lim\inf a_n\).

b. Give an example of a metric space \(X\) and a Cauchy sequence in \(X\) that does not converge.

c. Give an example of a bounded function on \([0, 1]\) that is not Riemann integrable on \([0, 1]\).

d. Give two examples, one of a convergent series and one of a divergent series, such that the root test is inconclusive for each.

e. Give an example of an ordered set \(S\) and a bounded subset \(E \subset S\) such that \(E\) has no greatest lower bound in \(S\).

f. Give an example of a sequence of continuous functions \(f_n : \mathbb{R} \to \mathbb{R}\) whose pointwise limit \(f : \mathbb{R} \to \mathbb{R}\) exists, but is discontinuous.

g. Give an example of a uniformly continuous function \(g : \mathbb{R} \to \mathbb{R}\) that is not differentiable on all of \(\mathbb{R}\).