MATH 521 (STOVALL). HOMEWORK 12.

Due Friday, 5/10, but you may turn it in Monday, 5/13 without penalty.

1. a. If $f : [a, b] \to \mathbb{R}$ is continuous, nonnegative, and has integral 0 on $[a, b]$, then $f(x) = 0$ for all $x \in [a, b]$.

   b. But if
   \[ f(x) = \begin{cases} 1, & x = c; \\ 0, & x \neq c, \end{cases} \]
   for some $c \in [a, b]$, then $f$ is Riemann integrable on $[a, b]$ and $\int_{a}^{b} f \, dx = 0$.

2. Show that if $f$ and $g$ are bounded Riemann integrable functions on the interval $[a, b]$, then $\sqrt{f^2 + g^2}$ is Riemann integrable as well.

   Hints: It suffices to prove this when $f, g \geq 0$ (why? how does this help?). You may use the fact (which follows from the triangle inequality in $\mathbb{R}^2$) that if $(M_1, M_2), (m_1, m_2) \in \mathbb{R}^2$, then
   \[ \sqrt{M_1^2 + M_2^2} - \sqrt{m_1^2 + m_2^2} \leq \sqrt{(M_1 - m_1)^2 + (M_2 - m_2)^2} \leq |M_1 - m_1| + |M_2 - m_2|. \]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function (i.e. assume that the $n$-th derivative $f^{(n)}$ is defined on all of $\mathbb{R}$). Let $a \in \mathbb{R}$ and let
   \[ R = \left( \limsup_{n \to \infty} \left| \frac{f^{(n)}(a)}{n!} \right|^{\frac{1}{n}} \right)^{-1}, \]
   where $0^{-1}$ is interpreted as $\infty$, and $\infty^{-1}$ is interpreted as 0.

   a. If $|x - a| < R$, then the Taylor series
   \[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \]
   converges absolutely, and if $|x - a| > R$, it diverges.

   b. For each $n \in \mathbb{N}$, let $M_n$ denote the maximum of $|f^{(n)}|$ on the interval $[a - R, a + R]$. If $\lim_{n \to \infty} \left| \frac{M_n}{n!} R^n \right| = 0$, then
   \[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \]
   for every $x \in (a - R, a + R)$.

One more on next page.
4. But DANGER!!! Define
\[ f(x) = \begin{cases} 
    e^{-\frac{1}{x}}, & x > 0 \\
    0, & x \leq 0. 
\end{cases} \]

a. Assuming that \( f \) is a smooth function, show that all of its derivatives at 0 are equal to 0. Thus the Taylor series has an infinite radius of convergence at 0, but \( f \) is certainly not equal to its Taylor series for any \( x > 0 \).
Hint: Use induction.

b. Show that \( f \) is a smooth function on all of \( \mathbb{R} \). You may use any standard fact from 221, such as
\[ \frac{d}{dx} e^x = e^x, \quad \lim_{t \to \infty} r(t)e^{-t} = 0, \text{ for any rational function } r, \quad \ldots. \]
Hint: Use induction to show that
\[ f^{(n)}(x) = \begin{cases} 
    0, & x \leq 0 \\
    \frac{p_n(x)}{x^n} e^{-\frac{1}{x}}, & x > 0, 
\end{cases} \]
for some polynomial \( p_n \). The formula is certainly valid for \( n = 0 \). Assume its validity for some \( n \geq 0 \). For \( x < 0 \), \( f \) is constant on a neighborhood of \( x \), so \( f^{(n+1)}(x) = 0 \). For \( x > 0 \), use the chain and product rules. For \( x = 0 \), you will have to use the definition of the derivative; treat the two directions, \( h \searrow 0 \) and \( h \nearrow 0 \) separately.

For those of you using LaTeX, putting a ‘\,’ between the function and the \( dx \) makes things look less weird.