In each problem, unless otherwise specified, $X$ is a nonempty set equipped with a metric $d$.

1. Carefully prove the following, which was stated in class: If $E \subseteq X$, $x \in \overline{E}$ if and only if for every $r > 0$, $N_r(x) \cap E \neq \emptyset$.

2. Show that if $\alpha > 0$, 
   \[
   \tilde{d}(x, y) = \frac{d(x,y)}{\alpha + d(x,y)}
   \]
   also defines a metric on $X$. Show that $X$ is a bounded set with respect to $\tilde{d}$.

3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $Z = X \times Y$. Show that the following define metrics on $Z$:
   \[
   d_1[(x_1, y_1), (x_2, y_2)] = d_X(x_1, x_2) + d_Y(y_1, y_2)
   \]
   \[
   d_\infty[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.
   \]

4. (Variant of Problem 9 in Rudin.) If $E \subseteq X$, we define the interior of $E$, denoted $\hat{E}$, to be the set of interior points of $E$. Then (by the definition of openness), $E$ is open if and only if $E = \hat{E}$.
   a. Prove that $(\hat{E})^c = (E^c)$.
   b. Prove that $\hat{E}$ is always an open set.
   c. Prove that $\hat{E}$ equals the union of open sets contained in $E$.

5. If $E \subseteq X$, we define the boundary of $E$, denoted $\partial E$, to be the set of points $x \in X$ such that every neighborhood of $x$ contains at least one point in $E$ and at least one point in $E^c$.
   a. Show that: $\hat{E} = E \setminus \partial E$, $\overline{E} = E \cup \partial E$.
   b. Show that $\partial E = \overline{E} \cap E^c = (\hat{E} \cup (E^c)\hat{o})^c$. Use this to show that $\partial E$ is closed.

6. Consider $\mathbb{R}$ with the usual metric. Carefully show that $\overline{\mathbb{Q}} = \mathbb{R}$, $\hat{\mathbb{Q}} = \emptyset$, and $\partial \mathbb{Q} = \mathbb{R}$. (Note: You will need to show that every neighborhood of any rational number contains an irrational number.) Thus a set and its interior may have different closures, a set and its closure may have different interiors, and a set and its boundary may have different boundaries.

7. Let $E, F \subseteq X$. Prove the following.
   a. $(E \cap F)^\hat{o} = \hat{E} \cap \hat{F}$.
   b. $(E \cup F)^\hat{o} \supseteq \hat{E} \cup \hat{F}$.
   c. $(E \cap F) \subseteq \overline{E} \cap \overline{F}$.
   d. $(E \cup F) = \overline{E} \cup \overline{F}$.
Honors problems

1. If \( x \in X \) and \( E \subseteq X \), define \( d(x, E) = \inf_{y \in E} d(x, y) \). If \( r > 0 \), let
   \[
   N_r(E) = \{ x \in E : d(x, E) < r \}.
   \]
   a. Show that \( N_r(E) \) is open and that \( \overline{E} = \bigcap_{r > 0} N_r(E) \).
   b. Let \( x, y \in X \) and \( E \subseteq X \). Show that \( d(x, E) \leq d(x, y) + d(y, E) \).

2. If \( A, B \subseteq X \), define
   \[
   D(A, B) = \sup\{ d(a, B) : a \in A \} + \sup\{ d(b, A) : b \in B \}.
   \]
   Let \( \mathcal{F} \) be the collection of closed and bounded subsets of \( X \). Show that \( D \) maps \( \mathcal{F} \times \mathcal{F} \) into \( \mathbb{R} \), that is, that \( D(A, B) < \infty \) whenever \( A \) and \( B \) are closed and bounded. Which of the metric space axioms does \( (\mathcal{F}, D) \) satisfy for general \( X \)? In each case, prove or give a counter-example.