In each problem, unless otherwise specified, subsets of \( \mathbb{R}^k \) are equipped with the Euclidean metric, and \( X \) is a nonempty set equipped with a metric \( d \).

1. Which of the following are compact? connected? Briefly justify your answers.
   a) \( \{ x \in \mathbb{R}^n : 1 \leq |x| \leq 2 \} \).
   b) \( \mathbb{N} \)
   c) \( \mathbb{Q} \cap [0, 1] \)

2. a) Let \( E \subseteq X \). Prove that \( E \) is not connected if and only if we can write \( E \subseteq A \cup B \) where \( A \cap E \neq \emptyset \), \( B \cap E \neq \emptyset \), \( A \cap B = \emptyset \), and \( A \) and \( B \) are both open.
   b) Use this to show that the metric space \( X \) is connected if and only if the only subsets of \( X \) that are both open and closed are \( X \) and \( \emptyset \).
   (Note, you may assume part a in proving b even if you do not know how to prove a.)

3. Show that the following sets are not compact by exhibiting an open cover with no finite subcover.
   a. \( E = \{ \frac{1}{n} : n \in \mathbb{N} \} \)
   b. \( E = \mathbb{Q} \cap [0, 1] \)
   c. Any infinite set equipped with the discrete metric.

4. Show that if the subset \( E \subseteq X \) is connected and contains more than 1 point, then \( E \) contains no isolated points. (Hence a connected set must contain one, zero, or infinitely many points.)

5. Prove the following.
   a. \( \lim_{n \to \infty} \left[ \sqrt{n^2 + 1} - n \right] = 0 \).
   b. \( \lim_{n \to \infty} \left[ \sqrt{n^2 + n - n} \right] = \frac{1}{2} \).

6. Prove that finite unions and arbitrary intersections are compact. In other words,
   a) Let \( F_1, \ldots, F_N \) be compact subsets of \( X \). Show that \( \bigcup_{n=1}^N F_n \) is compact.
   b) Let \( \{ F_\alpha \}_{\alpha \in A} \) be a collection of compact sets (for some index set \( A \)). Show that \( \bigcap_{\alpha \in A} F_\alpha \) is compact.
Honors problems

1. Show that a connected set containing at least two points must be uncountable.

2. Problems 23 and 24 of Chapter 2 in Rudin. (*Separable* is defined in problem 22.)