UNIFORM ESTIMATES FOR FOURIER RESTRICTION TO POLYNOMIAL CURVES IN $\mathbb{R}^d$

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ABSTRACT. We prove uniform $L^p \to L^q$ bounds for Fourier restriction to polynomial curves in $\mathbb{R}^d$ with affine arclength measure, in the conjectured range.

1. Introduction

In this article, we consider the problem of restricting the Fourier transform of an $L^p$ function on $\mathbb{R}^d$ to a curve $\gamma : \mathbb{R} \to \mathbb{R}^d$. It has been known for a number of years that when the curve is equipped with Euclidean arclength measure, the $p$ and $q$ such that this defines a bounded operator from $L^p$ to $L^q$ must depend on the maximal order of vanishing of the torsion $L_\gamma := \det(\gamma', \ldots, \gamma^{(d)})$. However, when the curve is equipped with affine arclength measure, $\lambda_\gamma \, dt = |L_\gamma|^{2/(d+2)} \, dt$, any degeneracies of curvature are mitigated and the known torsion-dependent obstructions vanish.

Because the sharp $L^p(dx) \to L^q(\lambda_\gamma \, dt)$ estimates for the restriction operator are completely invariant under affine transformations of $\mathbb{R}^d$ and reparametrizations of $\gamma$, there has been considerable interest (such as [2, 3, 4, 13, 16, 18, 19, 27]) in the question of whether such bounds hold uniformly over certain large classes of curves. This is part of a broader program ([8, 11, 14, 24, 26] and many others) to determine whether curvature-dependent bounds for various operators arising in harmonic analysis can be generalized, uniformly, by the addition of appropriate affine arclength or surface measures.

In the case of Fourier restriction to curves, the only known obstruction to $L^p(dx) \to L^q(\lambda_\gamma \, dt)$ boundedness is oscillation. The example $(t, e^{-1/t} \sin(t^{-k}))$, $0 < t < 1$ is due to Sjölin ([27]); an example with nonvanishing torsion is a line of irrational slope on the two-torus in $\mathbb{R}^3$. Motivated by this, a natural question, suggested by Dendrinos–Wright in [16], is whether there hold restriction estimates with constants that are uniform over the class of polynomial curves of any fixed degree. This question has been settled in the affirmative in dimension two [27]. In higher dimensions, the result has been proved for general polynomial curves in a restricted range [4, 16], and for monomial [2, 17] and ‘simple’ [4] polynomial curves in the full range. Our main theorem settles this question in the remaining conjectured cases.

Theorem 1.1. For each $N, d$, and $(p, q)$ satisfying

$$p' = \frac{d(d+1)}{2} q, \quad q > \frac{d^2+d+2}{d+2},$$

(1.1)

there exists a constant $C_{N,d,p}$ such that for all polynomials $\gamma : \mathbb{R} \to \mathbb{R}^d$ of degree less than or equal to $N$,

$$\|\hat{f}(\gamma(t))\|_{L^q(\lambda_\gamma \, dt)} \leq C_{N,d,p} \|f\|_{L^p(dx)},$$

(1.2)
for all Schwartz functions $f$.

This is sharp. A simple scaling argument shows that (1.2) can only (provided $\lambda_\gamma \neq 0$) hold if $p' = \frac{d(d+1)}{2} q$, and Arkhipov–Chubarikov–Kuratsuba proved in [1] that the restriction $q > \frac{d^2+d+2}{d^2+2d}$ is also necessary. At the endpoint $q_d = \frac{d^2+d+2}{d^2+2d} (at which point $p = q$), there are some cases where the corresponding restricted strong type bound is known [2, 4]; we do not address the endpoint case here.

The parametrization and affine invariance of (1.2) make the affine arclength measure a natural object of study. This aptness is underscored by the facts that it is essentially the largest positive measure such that the above $L^p \to L^q$ bounds can hold ([25], which also has an interesting geometric perspective), and in the case of a compact nondegenerate curve, the interpolation of the above estimates with elementary ones can be used to deduce essentially all valid $L^p \to L^q$ inequalities [1]. In the general polynomial case, we will show that Theorem 1.1 and an interpolation argument imply an extension into Lorentz spaces, as well as the full range of estimates for the unweighted operator. Let $K_{\min}$ equal the maximum order of vanishing of $L_\gamma$ on $\mathbb{R}$ and let $K_{\max}$ equal the degree of $L_\gamma$. Let $N_* = K_\gamma + \frac{d^2+d}{2}$. 

**Corollary 1.2.** Let $d \geq 3$ and let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a polynomial curve, with $L_\gamma \neq 0$. Define $d_\gamma(t) = \text{dist}(t, Z_\gamma)$, where $Z_\gamma$ is any finite set containing the complex zeros of $L_\gamma$. Then for all $1 < p < \frac{d^2+d+2}{d^2+2d}$ and $q \leq \frac{2p'}{d(d+1)}$,

$$\| \hat{f}(\gamma(t)) \|_{L^q(dt)} \| d_\gamma(t)^{-\gamma} \|_{L^{p}(dt)} \leq C_{p,q,\deg \gamma, \#Z_\gamma} \| f \|_{L^p(dx)}. \quad (1.3)$$

Furthermore, the unweighted operator satisfies

$$\| \hat{f}(\gamma(t)) \|_{L^q(dt)} \leq C_{p,q,\gamma} \| f \|_{L^p(dx)}, \quad (1.4)$$

if and only if $1 < p < \frac{d^2+d+2}{d^2+2d}$, and either $p \leq q$ and $N_{\min}q \leq p' \leq N_{\max}q$, or $p > q$ and $N_{\min} < p' < N_{\max}q$.

The use of the weight $d_\gamma$ and the resulting uniformity in (1.3) seem to be new, but the estimate and its proof are inspired by an argument in [18]; (1.4) sharpens and makes global certain estimates appearing in [12, 18, 19]. These bounds are valid even for $q < 1$ in the given range. The dependence of the constant in (1.4) on $\gamma$ is unavoidable because of the lack of affine and parametrization invariance. Related estimates in the context of generalized Radon transforms will also appear in [15]. We note that the techniques used in our proof could also be used to obtain the full range of estimates (necessarily nonuniform) for restriction with Euclidean arclength measure, but the exponents would be a bit more complicated.

**Prior results.** The problem of obtaining uniform bounds for restriction with affine arclength dates back to the 70s, when Sjölin [27] proved a sharp restriction result that is completely uniform over the class of convex plane curves. This result implies the two dimensional version of Theorem 1.1 by the triangle inequality.

In higher dimensions, the first results are due to Prestini in [22, 23], who proved restriction estimates for nondegenerate curves in a restricted range, just off the sharp line. In [12], Christ extended Prestini’s result to the sharp line and the range $q \geq \frac{d^2+2d}{d^2+2d}$, and obtained new estimates for unweighted restriction to certain degenerate curves. Shortly thereafter, Drury [17] extended Christ’s result to the full range ($q > \frac{d^2+2d}{d^2+2d}$) for nondegenerate curves.
For degenerate curves, the development was somewhat slower. In [18, 19], Drury–Marshall ultimately established sharp, global bounds in the so-called Christ range and local estimates in the interior of the conjectured region for restriction to monomial curves (with arbitrary real powers). It was not for another twenty years, when Bak–Oberlin–Seeger [2] proved (1.2) for monomial curves in the range (1.1), that any sharp estimates were known beyond the Christ range for any flat curves; [2] also established the endpoint (restricted strong type) estimate for nondegenerate curves. Very shortly thereafter, Dendrinos–Wright [16] posed the problem considered here and also proved Theorem 1.1 in the Christ range. In [13], Dendrinos–Müller proved that the estimates for monomial curves from [2] are stable under sufficiently small perturbations. Shortly thereafter, Bak–Oberlin–Seeger [4] established uniform bounds in the full range (1.1) for ‘simple’ polynomial curves, i.e. curves of the form $(t, t^2, \ldots, t^{d-1}, P(t))$, and slightly extended the Dendrinos–Wright range for general polynomial curves. Our result is new in the remaining cases.

Outline of proof. Though we build on much of the above-mentioned literature (especially [12, 16, 17, 18]), we take a new approach for the degenerate case, particularly compared to the recent [2, 4, 13], by using a dyadic decomposition according to torsion size, coupled with a square function estimate.

We begin in Section 2 by considering the problem of restricting $\hat{f}$ to a segment along $\gamma$ that has roughly constant torsion. By a scaling and compactness argument, together with an induction argument from [17], we establish uniform estimates along this segment without facing the significantly more delicate task of performing real interpolation in the presence of the affine arclength (cf. [2, 13]). We close with a more detailed discussion of the techniques in other recent articles.

In order to use the estimates from Section 2, we must decompose the operator according to the size of the torsion, while not ruining our chances of being able to put the pieces back together later. This we do in Section 3 by means of a uniform square function estimate for the extension operator. The heuristic behind this is the (false) assertion that if the torsions of two points on the curve are at different scales, then the points themselves must be at different frequency scales.

We complete the proof of Theorem 1.1 in Section 4. Using an argument inspired by the recent success of bilinear and multilinear approaches to Fourier restriction to hypersurfaces, we reduce matters to proving a $\frac{d(d+1)}{2}$-linear extension estimate, which exhibits decay when the arguments live at different torsion scales on the curve. The core of the argument is a variant of Christ’s multilinear estimate, together with interpolation with our bounds from Section 2.

In Section 5, we prove the corollary.

It would be interesting to see whether these ideas could be used to obtain more uniform bounds for restriction to sufficiently smooth finite type curves, or whether some of the simplifications here could lead to progress on the endpoint restricted strong type bounds in the general polynomial case. Somewhat more broadly, our approach of transferring estimates from the non-degenerate to the degenerate case, particularly the use of the square function estimate and multilinear estimates with decay, may be useful in establishing bounds for Fourier restriction to degenerate submanifolds of higher dimension.

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The dual formulation and other notation. In proving Theorem 1.1, we will focus on the corresponding extension problem. Fix a polynomial \( \gamma : \mathbb{R} \to \mathbb{R}^d \). Define the weighted and unweighted extension operators

\[
\mathcal{E}_\gamma f(x) = \int_\mathbb{R} e^{ix\gamma(t)} f(t) \lambda_\gamma(t) \, dt \quad \mathcal{F}_\gamma f(x) = \int_\mathbb{R} e^{ix\gamma(t)} f(t) \, dt.
\]

Theorem 1.1 is equivalent to showing that

\[
\|\mathcal{E}_\gamma f\|_{L^q(dx)} \leq C_{d,p,N} \|f\|_{L^p(\lambda_\gamma dt)},
\]

for all \((p, q)\) in the range

\[
q = \frac{d(d+1)}{2} p', \quad q > \frac{d^2+d+2}{2}.
\]

An affine transformation is a map of the form \( Ax = Mx + x_0 \), where \( M \) is a \( d \times d \) matrix and \( x \in \mathbb{R}^d \). We will denote by \( \det A \) the determinant of the corresponding linear map, \( \det A = \det M \).

As usual, if \( B_1 \) and \( B_2 \) are two non-negative quantities, we will write \( B_1 \lesssim B_2 \) if \( B_1 \leq CB_2 \) for some innocuous constant \( C \). These constants will be allowed to change from line to line.

2. Uniform local restriction

The bulk of this section will be devoted to a proof of the following theorem. We will conclude the section with a detailed comparison of our approach with the recent literature.

**Theorem 2.1.** Fix \( d \geq 2 \), \( N \), and \((p, q)\) satisfying (1.6). For every interval \( I \subset \mathbb{R}^d \) and every degree \( N \) polynomial \( \gamma : \mathbb{R} \to \mathbb{R}^d \) satisfying

\[
0 < C_1 \leq |\det(\gamma'(t), \ldots, \gamma^{(d)}(t))| \leq C_2, \quad t \in I,
\]

we have the extension estimate

\[
\|\mathcal{E}_\gamma (\chi_I f)\|_{L^q} \leq C_{d,N,p,C_2} \|f\|_{L^p(\lambda_\gamma)}.
\]

We will use the notation:

\[
\gamma(t) = \det(\gamma'(t), \ldots, \gamma^{(d)}(t)), \quad J_\gamma(t_1, \ldots, t_d) = \det(\gamma'(t_1), \ldots, \gamma'(t_d)),
\]

\[
v(t_1, \ldots, t_d) = c_d \prod_{1 \leq i < j \leq d} (t_j - t_i) \text{ is the Vandermonde determinant.}
\]

Since \( L_\gamma \) is a polynomial of degree less than \( Nd \), we may write \( I = \bigcup_{j=1}^{C N d \log(C_2)} I_j \) where

\[
\frac{1}{2} C \leq |L_\gamma(t)| \leq 2C, \quad t \in I_j.
\]

By the triangle inequality, it suffices to prove (2.2) with \( I \) replaced by one of the \( I_j \). Utilizing the affine and parametrization invariances, we may assume that \( C = 1 \) and \( I = [-1, 1] \). In other words, we may assume that \( \gamma \) satisfies

\[
\frac{1}{2} \leq |L_\gamma(t)| \leq 2, \quad t \in [-1, 1],
\]

\[
(2.3)
\]
and we want to prove
\[ \|\mathcal{E}_\gamma(\chi_I f)\|_{L^q} \leq C_{d,N,p} \|f\|_{L^p(\Lambda_\gamma)}, \tag{2.4} \]
where \( I = [-1,1] \) and \((p, q)\) satisfies (1.6).

**Lemma 2.2.** If \( \gamma : \mathbb{R} \to \mathbb{R}^d\) is a degree \( N \) polynomial satisfying (2.3), there exists an affine transformation \( A \) with \( \det A = 1 \) and \( \|A\gamma\|_{C^N([-1,1])} \leq C_{N,d} \).

**Proof.** Let \( A \) denote the affine transformation
\[ Ax = [\gamma'(0) \cdots \gamma^{(d)}(0)]^{-1}(x - \gamma(0)). \]
Then \( \frac{1}{2} \leq \det A \leq 2 \) by (2.3), so (replacing \( A \) with \( (\det A^{-1})A \) if needed) the lemma will be proved if we can show that \( \|A\gamma\|_{C^N([-1,1])} \leq C_{N,d} \).

To simplify the notation, we assume henceforth that \( A\gamma = \gamma \). Thus
\[ \frac{1}{2} \leq L_\gamma(t) \leq 4, \quad t \in [-1,1], \quad \gamma(0) = 0, \quad \gamma^{(j)}(0) = e_j, \quad 1 \leq j \leq d. \tag{2.5} \]
By Taylor’s theorem, it suffices to show that \( |\gamma^{(j)}(0)| \leq C_{d,N} \) for all \( j \geq 1 \), and hence by (2.5), it suffices to show that
\[ |\det(\gamma^{(n_1)}(0), \ldots, \gamma^{(n_d)}(0))| \leq C_{d,N}, \tag{2.6} \]
for all \( n_1 < \cdots < n_d \); this is because \([\gamma'(0), \ldots, \gamma^{(d)}(0)]\) is the identity, so we can replace any column by \( \gamma^{(n)}(0) \) to pick out the coefficient we want.

Suppose that the lemma is false. Then there exists a sequence of polynomials \( \gamma_n \) satisfying (2.5) such that
\[ \max_{1 \leq n_1 < \cdots < n_d \leq N} |\det(\gamma^{(n_1)}_n(0), \ldots, \gamma^{(n_d)}_n(0))| \to \infty. \tag{2.7} \]
Let \( (\delta_n) \), \( 0 < \delta_n < 1 \), be a sequence, to be determined in a moment. Define rescaled curves by
\[ \Gamma_n(t) = (\delta_n^{-1}\gamma_{n,1}(\delta_n t), \ldots, \delta_n^{-d}\gamma_{n,d}(\delta_n t)). \]
Observe that \( \Gamma_n \) obeys (2.5) on \([-\delta_n^{-1}, \delta_n^{-1}]\) and that if \( n_1 + \cdots + n_d > \frac{d(d+1)}{2} \),
\[ |\det(\Gamma^{(n_1)}_n(t), \ldots, \Gamma^{(n_d)}_n(t))| \leq \delta_n |\det(\gamma^{(n_1)}(\delta_n t), \ldots, \gamma^{(n_d)}(\delta_n t))|. \]
By (2.7), for each \( n \) sufficiently large, we may choose \( 0 < \delta_n < 1 \) so that
\[ \max_{1 \leq n_1 < \cdots < n_d \leq N} |\det(\Gamma^{(n_1)}_n(0), \ldots, \Gamma^{(n_d)}_n(0))| = 5; \tag{2.8} \]
(2.7) further implies that \( \delta_n \to 0 \). Passing to a subsequence, there exists a single \( d \)-tuple \( 1 \leq n_1 < \cdots < n_d \leq N \) such that
\[ |\det(\Gamma^{(n_1)}(0), \ldots, \Gamma^{(n_d)}(0))| = 5, \quad \text{for all } n. \tag{2.9} \]
By (2.5), \( n_1 + \cdots + n_d > \frac{d(d+1)}{2} \).

On the other hand, by (2.8), (2.5), and the observation after (2.6), \( |\Gamma^{(j)}(0)| \leq C_{N,d} \) for all \( j \). In other words, all of the coefficients of \( \Gamma_n \) are bounded. Thus after passing to a subsequence, there exists a limit, \( \Gamma_n \to \Gamma \) (in the metric space of polynomial curves of degree at most \( N \)). By (2.5) and the fact that \( \delta_n \to 0 \),
\[ \frac{1}{4} \leq |L_\Gamma(t)| \leq 4, \quad t \in \mathbb{R}. \]
So by (2.5), \( L_\Gamma(t) \equiv 1 \), since \( \Gamma \) is a polynomial. This implies that
\[ \Gamma(t) = (t, \frac{1}{2}t^2, \ldots, \frac{1}{4d}t^d) \]
(no affine transformation is necessary by (2.5)). But by (2.9),

$$|\det(\Gamma^{(n_1)}(0), \ldots, \Gamma^{(n_d)}(0))| = 5,$$

a contradiction. This completes the proof.

Theorem 2.1 almost follows from Lemma 2.2 by a result of Drury (Theorem 2 of [17]), but some additional uniformity is needed. The first step is to obtain estimates for the offspring curves of $\gamma$.

**Lemma 2.3.** Fix $d \geq 2$ and $N$. There exists a constant $c_d > 0$ and a decomposition

$$[-1, 1] = \bigcup_{j=1}^{M_{d,N}} I_j$$

into disjoint intervals such that the conclusions below hold for every $I = I_j$ and every degree $N$ polynomial $\gamma : \mathbb{R} \to \mathbb{R}^d$ satisfying

$$\frac{1}{2} \leq |L_\gamma(t)| \leq 2, \quad t \in [-1, 1]. \quad (2.10)$$

If $K \geq 1$ and $(h_1, \ldots, h_K) \in \mathbb{R}^K$, the curve defined by $\gamma_h(t) = \frac{1}{K} \sum_{j=1}^K \gamma(t + h_j)$ satisfies the following on the interval $I_h := \gamma^{K}_{j=1}(I - h_j)$:

$$|J_{\gamma_h}(t_1, \ldots, t_d)| \geq c_d \prod_{j=1}^d |L_{\gamma_h}(t_j)|^{\frac{1}{2}} \prod_{1 \leq i < j \leq d} |t_j - t_i|$$(2.11)

$$c_d \leq |L_{\gamma_h}(t)| \leq c_d^{-1}.$$ (2.12)

In [17], an argument is given to prove an analogous lemma with the weaker hypothesis that $\gamma$ is $C^d$. In that more general case, the uniformity we need is impossible, so we give a detailed proof of Lemma 2.3 (via a different argument) here.

**Proof.** Fix a curve $\gamma$ satisfying the hypotheses of the lemma and let $t_0 \in [-1, 1]$. Let $\delta > 0$ be a sufficiently small constant, depending only on $d$ and $N$ and to be determined in a moment. We will show that the conclusions of the lemma hold on $I := [t_0 - \delta, t_0 + \delta]$. This is sufficient.

By Lemma 2.2, by reparametrizing and performing an affine transformation on $\gamma$ if necessary, we may assume that $t_0 = 0$, $\gamma(0) = 0$, $\gamma^{(j)}(0) = e_j$, $1 \leq j \leq d$, and $|\gamma^{(j)}(0)| \leq C_{N,d}$ for all $j \geq 1$. Therefore

$$\gamma(t) = (t, \frac{1}{2}t^2, \ldots, \frac{1}{d!}t^d) + \tilde{\gamma}(t),$$

where $\tilde{\gamma}$ is a degree $N$ polynomial with $\tilde{\gamma}^{(j)}(0) = 0$ for $0 \leq j \leq d$ and $|\tilde{\gamma}^{(j)}(0)| \leq C_{N,d}$ for all $j$.

Now fix $K \geq 1$ and $h \in \mathbb{R}^K$. We can translate $h$, which just shifts $I_h$, and reorder its components with impunity, so without loss of generality, $0 = h_1 < h_2 < \cdots < h_K$. If $I_h = \emptyset$, the conclusions of the lemma are trivial, so we may assume that $h_K < 2\delta$.

Define a matrix $A_h$ by

$$(A_h)_{ij} = \begin{cases} 0, & \text{if } i < j; \\ \frac{1}{K} \sum_{k=1}^K \frac{h_{i-j}^{(j)}}{(i-j)!}, & \text{if } i \geq j. \end{cases}$$
In particular, $A_h$ is lower triangular with ones on the diagonal, so it is invertible. Define
\[ \Gamma_h(t) = A_h^{-1}(\gamma_h(t) - \frac{1}{R} \sum_{j=1}^{K} (h_j, \frac{1}{R} h_j^2, \ldots, \frac{1}{R^d} h_j^d)), \quad \tilde{\Gamma}_h(t) = A_h^{-1}\tilde{\gamma}_h(t). \]
Since
\[ A_h(t, \ldots, \frac{1}{R} t^d) = \frac{1}{R} \sum_{k=1}^{K} (t + h_j, \ldots, \frac{1}{R} (t + h_j)^d) - \frac{1}{R} \sum_{k=1}^{K} (h_j, \ldots, \frac{1}{R^d} h_j^d), \]
we have
\[ \Gamma_h(t) = (t, \ldots, \frac{1}{R} t^d) + \tilde{\Gamma}_h(t). \]
Since $\|A_h - I\| \lesssim_d \delta$, for $\delta$ sufficiently small (depending only on $d$), $\|A_h^{-1} - I\| \lesssim_d \delta$. Furthermore, because $\tilde{\gamma}^{(j)}(0) = 0$ for $1 \leq j \leq d$ and $|\tilde{\gamma}^{(j)}(0)| \leq C_{d,N}$ for all $j$, $|\tilde{\gamma}^{(j)}_h(0)| \leq C_{d,N} \delta^{d+1-j}$ for $1 \leq j \leq d$ and $|\tilde{\gamma}^{(j)}_h(0)| \leq C_{d,N}$ for all $j$. Combining these bounds
\[ |\tilde{\Gamma}_h^{(j)}(t)| \leq C_{d,N} \delta^{d+1-j}, \quad |\tilde{\Gamma}_h^{(k)}(t)| \leq C_{d,N}, \quad \text{for all} \quad 1 \leq j \leq d \leq k, \quad t \in [-\delta, \delta]. \]
From this and multilinearity of the determinant,
\[ L_{\gamma_h}(t) = L_{\Gamma_h}(t) = 1 + O_{d,N}^{\delta}), \]
which implies (2.12) for $\delta$ sufficiently small.

Since $J_{\Gamma_h}$ is an antisymmetric polynomial, for any $s \in \mathbb{R}$, we can write
\[ J_{\Gamma_h}(t_1, \ldots, t_d) = P_{\Gamma_h}(t_1, \ldots, t_d) \prod_{1 \leq i < j \leq d} (t_j - t_i) \]
\[ = c_d \sum_{\sigma \in S_d} \text{sgn}(\sigma) P_{\Gamma_h}(t_{1}, \ldots, t_{d})(t_{\sigma(2)} - s)(t_{\sigma(3)} - s)^2 \cdots (t_{\sigma(d)} - s)^{d-1}, \]
where $P_{\Gamma_h}$ is a symmetric polynomial of degree less than $Nd$ and $S_d$ denotes the symmetric group on $d$ letters. Using the second line of (2.13), we differentiate $J_{\Gamma_h}$ term-by-term and evaluate at $(s, \ldots, s)$:
\[ P_{\Gamma_h}(s, \ldots, s) = b_d \partial_{d-1}^{d-1} \partial_{d-2}^{d-2} \cdots \partial_1^{d-1} \big|_{t=(s, \ldots, s)} J_{\Gamma_h}(t); \]
this is because the derivatives must fall on the Vandermonde term. (Here, $b_d$ is a dimensional constant which will be allowed to change from line to line.) On the other hand,
\[ \partial_{d-1}^{d-1} \cdots \partial_1^{d-1} \big|_{t=(s, \ldots, s)} J_{\Gamma_h}(t) = L_{\Gamma_h}(s), \]
so
\[ P_{\Gamma_h}(s, \ldots, s) = b_d L_{\Gamma_h}(s). \]

In addition, because $|P_{\Gamma_h}(s)| \leq C_{N,d}$ for all $s \in [-\delta, \delta]$ and all $j \geq 1$, the derivatives of $P_{\Gamma_h}$ also satisfy
\[ |\partial^{\alpha} P_{\Gamma_h}(s, \ldots, s)| \leq C_{d,N}, \]
for all multiindices $\alpha$ and all $s \in [-\delta, \delta]$. Therefore,
\[ P_{\Gamma_h}(t) = P_{\Gamma_h}(s, \ldots, s) + O_{d,N}(\delta) = b_d L_{\Gamma_h}(s) + O_{d,N}(\delta) = b_d + O_{d,N}(\delta), \]
for all $(t_1, \ldots, t_d) \in [-\delta, \delta]^d$ and $s \in [-\delta, \delta]$. Combining this with (2.13), we obtain (2.11).

This completes the proof of the proposition. \qed
Drury’s induction argument from [17] completes the proof of the theorem. (A previous draft had used the method of [2, 13]; Spyros Dendrinos kindly pointed out that the estimates in Lemmas 2.2 and 2.3 meant that Drury’s approach, which is somewhat more direct, could be used.) The base case is the trivial observation that
\[
\|E_\gamma f\|_{L^\infty} \leq \|f\|_{L^1(\lambda_\gamma)},
\]  
(2.14)
for any function \( f \) and any \( C^d \) curve \( \gamma \). Our hypothesis is the statement that for some \( 1 \leq p < \frac{d^2 + d + 2}{2} \), there exists a constant \( C_{d,p} \) such that
\[
\|E_\gamma (\chi_I f)\|_{L^{\frac{d(d+1)p}{d+2}}} \leq C_{d,p} \|f\|_{L^p(\lambda_\gamma)},
\]  
(2.15)
for all \( K \geq 1 \), \( h \in \mathbb{R}^K \).

We want to increase \( p \) (decreasing \( p \) is easy by interpolation with (2.14)). The inductive step is the following.

**Lemma 2.4 ([17]).** If the hypothesis (2.15) is valid for some \( 1 \leq p_0 < \frac{d^2 + d + 2}{2} \), then it is also valid for all \( p \geq 1 \) satisfying the inequality
\[
\frac{d}{p} > \frac{2}{d+2} + \frac{(d-2)p_0^{-1}}{d+2}.
\]

For the convenience of the reader and to make it easier to describe the background in more detail, we sketch the argument below. Complete details are given in [17] in the case \( \gamma = (t, t^2, t^3) \). We recall the notation
\[
v(h) = c_d \prod_{1 \leq i < j \leq d} (h_j - h_i).
\]

**Sketch of proof.** Let \( \tilde{\gamma} \) be an offspring curve: \( \tilde{\gamma} = \gamma_h : \tilde{I} = I_h \to \mathbb{R}^d \) for some \( K \geq 1 \), \( h \in \mathbb{R}^K \). By virtue of (2.12), it suffices to bound the unweighted operator
\[
\mathcal{F}_{\tilde{\gamma}} f(x) = \int_{\tilde{I}} e^{ix\tilde{\gamma}(t)} f(t) \, dt.
\]
We denote by \( \mu \) the measure defined by
\[
\mu(\phi) = \int_{\tilde{I}} \phi(\tilde{\gamma}(t)) \, f(t) \, dt.
\]
Since
\[
\|\mathcal{F}_{\tilde{\gamma}} f\|_{L^p} = \|\mu^d\|_{\frac{p}{p-1}}^{\frac{1}{p}},
\]  
(2.16)
we are interested in the \( d \)-fold convolution of \( \mu \).

If \( g(\xi) = \mu * \cdots * \mu(\xi) \), a computation shows that
\[
g\left(\frac{1}{d}(\tilde{\gamma}(t_1) + \cdots + \tilde{\gamma}(t_d))\right) = \frac{c_d}{J_{\tilde{\gamma}(t_1, \ldots, t_d)}} f(t_1) \cdots f(t_d).
\]
(We must change variables to do this. It is a consequence of (2.11) and (2.12) that \( (t_1, \ldots, t_d) \mapsto \sum \tilde{\gamma}(t_j) \) is one-to-one on \( \{ t \in I^d : t_1 < \cdots < t_d \} \). See e.g. [19, Section 3]. Alternatively, \( C_{d,N} \)-to-one follows from Bezout’s theorem.)

For \( h = (h_1, h') \in \{0\} \times \mathbb{R}^{d-1} \), we define
\[
G(t; h) = g\left(\frac{1}{d}(\tilde{\gamma}(t + h_1) + \cdots + \tilde{\gamma}(t + h_d))\right).
\]
By a change of variables in the \( \xi \) variable,
\[
\tilde{g}(x) = C_d \int_{\mathbb{R}^{d-1}} \int_{I_h} e^{ix\tilde{\gamma}_h(t)} g(\tilde{\gamma}_h(t)) |J_{\tilde{\gamma}}(t + h_1, \ldots, t + h_d)|^{-1} \, dt \, dh',
\]
where here and for the remainder of the section, we use the convention that $h_1 = 0$. By Plancherel, (2.11), and (2.12),
\[
\| \hat{g} \|_{L^2} \leq C_d \| G \|_{L^p_{\gamma}(L^\infty_{\gamma}|v(h)|^{-1})},
\]
and by (2.15) (plus interpolation to decrease $p$) and the integral form of Minkowski’s inequality,
\[
\| \hat{g} \|_{L^2} \leq C_{d,p} \| G \|_{L^p_{\gamma}(L^\infty_{\gamma}|v(h)|^{-1})}, \quad 1 \leq p \leq p_0, \quad q = \frac{d(d+1)}{2} p_0'.
\]
Thus by interpolation,
\[
\| \hat{g} \|_{L^p} \leq C_{d,a,b} \| G \|_{L^p_{\gamma}(L^\infty_{\gamma}|v(h)|^{-1})}, \tag{2.17}
\]
for all $(a^{-1}, b^{-1})$ in the triangle with vertices $(1, 1), (1, p_0^{-1}), (\frac{1}{2}, \frac{1}{2})$ and $c$ satisfying
\[
\frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} + \frac{d(d+1)}{2} c^{-1} = \frac{d(d+1)}{2}.
\]
A computation (again using (2.11) and (2.12)) shows that
\[
\| G \|_{L^2_{\gamma}(L^\infty_{\gamma}|v(h)|^{-1})} \sim_{d,a,b} \left( \int_{I_h} \| v(h) \|^{-(a^{-1})} \left( \int_{I_h} |f(t + h_1) \cdots f(t + h_d)|^b \, dt \right)^{\frac{1}{b}} \, dh \right)^{\frac{1}{a}}.
\]
Using this and the fact that $v(0, h') \in L^2_{h'}$, one can show that
\[
\| G \|_{L^p_{\gamma}(L^\infty_{\gamma}|v(h)|^{-1})} \leq \| f \|_{L^p_{u,1}}^{d}, \quad \text{for}
\]
\[
1 < a < \frac{d+2}{d}, \quad a \leq b < \frac{2a}{\pi+2-a}, \quad \frac{d}{p} = \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2}.
\]
Details may be found in [2, 17].

By (2.16), (2.17), (2.18), and some careful arithmetic,
\[
\| \mathcal{F}_g f \|_{L^q} \leq C_{d,q} \| f \|_{L^p_{u,1}}, \tag{2.19}
\]
for $p, q, a, b$ satisfying
\[
q = \frac{d(d+1)}{2} p', \quad \frac{d}{p} = \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2},
\]
\[
\frac{d}{\pi+2} < a^{-1} < 1, \quad b^{-1} \leq a^{-1}, \quad \frac{d+2}{a} - \frac{2}{b} < d, \quad (p_0 - 2) a^{-1} + p_0 b^{-1} \geq p_0 - 1.
\]
The point $(a^{-1}, b^{-1}) = (\frac{d}{\pi+2}, \frac{2}{\pi+2} + \frac{d-2}{(d+2)p_0})$ lies on the boundary of this region and satisfies
\[
\frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2} < \frac{d}{p_0}. \tag{2.20}
\]
Thus taking $(a^{-1}, b^{-1})$ slightly inside and using real interpolation,
\[
\| \mathcal{F}_g f \|_{L^q} \leq C_{d,p} \| f \|_{L^p_{u}}, \quad q = \frac{d(d+1)}{2} p', \quad \frac{d}{p} > \frac{2}{\pi+2} + \frac{d-2}{(d+2)p_0}.
\]
This completes the proof of the lemma and thus of Theorem 2.1.

**Comparison with prior work.** With these arguments in place, it is easier to put our result and our approach in context. Let $I$ be an interval and $\gamma : I \to \mathbb{R}^d$ be a $C^d$ curve. To obtain bounds in the Christ range, $q \geq \frac{d^2+2d}{2}$, it is not necessary to deal with the offspring curves; estimate (2.11) with $h = 0$ (which we will call the basic geometric inequality), together with some control on the growth of $L_\gamma$, is sufficient (see [12, 16, 19]). For general polynomial curves, the basic geometric inequality is not quite true globally, but in [16], Dendrinos–Wright proved a sufficiently uniform substitute (Lemma 3.1).

All known proofs beyond the Christ range rely on the method of offspring curves, and this method has seemed much more difficult when $L_\gamma$ is not roughly constant
as in (2.1). Two types of complications arise: geometric and analytic. On the
geometric front, (2.12) is simply not possible and must be replaced by \(|L_\gamma(t)| \gtrsim |L_{\gamma_h}(t)|, t \in I_{\gamma_h}\). Even with this adjustment, in practice it has been somewhat easier
to prove the basic geometric inequality than the appropriate analogue of Lemma 2.3
for those classes of curves (nondegenerate ones and sufficiently small perturbations
of monomials) for which both are known. On the analytic front, in the weighted
case, the real interpolation following (2.19) is more difficult (cf. [2]).

Now we turn to the perturbed monomial case. Let \(a_1 < \cdots < a_d\) be real numbers
and let \(\gamma(t) = (t^{a_1}\theta_1(t), \ldots, t^{a_d}\theta_d(t))\), with \(\theta_i \in C^d([0, 1])\)
and \(\theta_i(0) \neq 0\). Drury–Marshall proved the analogue of Lemma 2.3 in the case \(\theta_i \equiv 1\) (an omission in the
perturbed case is noted in [13]), and used this to obtain restriction estimates off
the sharp line (sharp in the Christ range). Finally, more than twenty years later,
Bak–Oberlin–Seeger overcame the analytic difficulties and proved the full range of
restriction estimates for these curves [2], again when \(\theta_i \equiv 1\). In [13], Dendrinos–
Müller proved that in the general case, there exists a constant \(\delta = \delta_\gamma > 0\) such that
the analogue of Lemma 2.3 holds on \(I = [0, \delta]\) and, arguing similarly to [2], showed
that this gives the full range of \(L^p \to L^q\) estimates for restriction to \(\gamma|_{[0, \delta]}\). It is not
claimed in [13], but this implies a non-uniform result for polynomials. Indeed, if
\(\gamma\) is a polynomial, then using the reparametrization \(t \mapsto t^{-1}\) near \(\pm \infty\), near every
point of \(\mathbb{R} \cup \{\pm \infty\}\), after an affine transformation, \(\gamma\) equals a small perturbation
of a monomial curve, and so the result follows by compactness of \(\mathbb{R} \cup \{\pm \infty\}\).

Unfortunately, the argument outlined above does not suggest an approach toward
proving uniform estimates, even in the polynomial case. This is not merely a techni-
cal issue. Dating back to Sjölin’s theorem on convex plane curves [27], a major goal
has been establishing uniform estimates, and in fact the chief motivation for study-
ing the polynomial case at all has been that it is very clear what the uniform result
Wright range in the general polynomial case by using a result from [30], and also
proved the full range of estimates for simple curves \(\gamma(t) = (t, \ldots, t^{d-1}, \phi(t))\).

There had been some evidence that obtaining the remaining uniform estimates
for polynomial curves might be substantially more difficult. The proof of the basic
geometric inequality for polynomials is long and highly nontrivial (it constitutes the
bulk of [16]), and by comparison with the proof of Proposition 8 in [13], the uniform
version seems substantially harder than the local version. (Roughly, the freedom to
choose \(\delta_\gamma\) to depend on \(\gamma\) seems to make the problem somewhat easier.) The task
of proving a sufficiently uniform global version of Lemma 2.3 for polynomials seems
potentially even more daunting than the basic geometric inequality, and because
the constant \(\delta_\gamma\) depends on the local behavior of the curve in a complicated way
(and because it is even more difficult to tease out the local behavior after applying
the affine transformations needed to make the curve locally monomial-like), the
arguments of [13] do not seem to offer a clear path forward.

We will avoid the above-mentioned geometric and analytic complications by
localizing to dyadic torsion scales. Our task for the next two sections is to show
that this localization is reasonable by recovering the global restriction estimate.

3. A uniform square function estimate

The following lemma is essentially due to Dendrinos–Wright in [16].
Lemma 3.1. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a polynomial of degree $N$, and assume that $L_{\gamma} \neq 0$. We may decompose $\mathbb{R}$ as a disjoint union of intervals,

$$
\mathbb{R} = \bigcup_{j=1}^{M_{N,d}} I_j,
$$

so that for $t \in I_j$,

$$
|L_{\gamma}(t)| \sim A_j |t - b_j|^{k_j}, \quad |\gamma_1'(t)| \sim B_j |t - b_j|^{\ell_j},
$$

and for all $(t_1, \ldots, t_d) \in I_j^d$,

$$
|J_{\gamma}(t_1, \ldots, t_d)| \gtrsim \prod_{j=1}^d |L_{\gamma}(t_j)|^{\frac{1}{2}} \prod_{1 \leq i < j \leq d} |t_j - t_i|,
$$

where for each $j$, $k_j$ and $\ell_j$ are integers satisfying $0 \leq k_j \leq dN$ and $0 \leq \ell_j \leq N$, and the centers $b_j$ are real numbers not contained in the interior of $I_j$. Furthermore, the map $(t_1, \ldots, t_d) \mapsto \sum_{j=1}^d \gamma_j(t_j)$ is one-to-one on $\{t \in I_j^d : t_1 < \cdots < t_d\}$. The implicit constants and $M_{N,d}$ depend on $N$ and $d$ only.

The main difficulty in proving this lemma is establishing (3.2). Fortunately, this has already been done in [16], and we will make no attempt to recap the lengthy argument. As for the rest, strictly speaking, Dendrinos–Wright prove this lemma without the estimate on $\gamma_1'$, but Lemma 3.1 may be obtained from their theorem in a straightforward manner. We briefly explain how this can be done.

The deduction of Lemma 3.1 from [16]. Two decomposition procedures are employed in [16].

For the first, given a polynomial $Q$ and interval $I$, the ‘D1’ procedure decomposes $\mathbb{R}$ into a union of $O(\deg Q)$ intervals, $I = \bigcup J$ such that on $J$, $|Q(t)| \sim A_j |t - b_j|^{k_j}$, where $b_j$ is the real part of a zero of $Q$ and $k_j$ is an integer with $0 \leq k_j \leq \deg Q$.

The second procedure is due to Carbery–Ricci–Wright in [10]. Given a polynomial $P$ and a center $b$, the ‘D2’ procedure decomposes $\mathbb{R}$ into a union of $O(\deg P)$ ‘gaps’ and $O(\deg P)$ ‘dyadic intervals.’ On a gap, $|P(t)| \sim A |t - b|^k$, for some integer $0 \leq k \leq \deg P$. On a dyadic interval $|t - b| \sim A$ for some constant $A$.

From [16], we know that it is possible to decompose $\mathbb{R} = \bigcup I$ in such a way that for each $I$, (3.2) holds on $I^d$. Performing the D1 procedure with $Q = L_{\gamma}$, we may additionally assume that $L_{\gamma}(t) \sim A_j |t - b_j|^{k_j}$ on $I$. We fix $I$ and perform the D2 decomposition with $P = \gamma_1'$ and $b = b_j$. If $G$ is a gap interval, the conclusions of the lemma hold on $G \cap I$. If $D$ is a dyadic interval, we perform the D1 decomposition with $Q = \gamma_1'$ on $D \cap I$. If $J \subset G \cap I$ is an interval resulting from this decomposition, $|\gamma_1'(t)| \sim B_j |t - b_j|^{k_j}$ and $|L_{\gamma}(t)| \sim A_j$ on $J$, so the conclusions of the lemma hold.

Our square function estimate is the following.

Proposition 3.2. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a polynomial of degree $N$, and assume that $L_{\gamma} \neq 0$. Let $\{I_j\}_{j=1}^{M_{N,d}}$ denote the collection of intervals from Lemma 3.1. Fix $j$. For $n \in \mathbb{Z}$, define

$$
I_{j,n} = \{t \in I_j : 2^n \leq |t - b_j| < 2^{n+1}\}.
$$
Then for each $(p,q)$ satisfying $q = \frac{d(d+1)}{2}p'$ and $\infty > q > \frac{d^2 + 2d}{2}$, $f \in L^p$, and $j$,
\[
\|\mathcal{E}_\gamma(\chi_{I_j} f)\|_{L^q(\mathbb{R}^d)} \lesssim \left\| \left( \sum_{n} |\mathcal{E}_\gamma(\chi_{I_{j,n}} f)|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^d)}.
\] (3.3)

The implicit constant depends only on $N,d,q$.

**Proof of Proposition 3.2.** If $k_j = 0$, the right side of (3.3) involves $O(1)$ values of $n$, so the inequality is trivial. We assume henceforth that $k_j > 0$.

By standard approximation arguments, we may assume that $f$ is supported in the union of finitely many of the $I_{j,n}$. Thus by Theorem 2.1, $\mathcal{E}_\gamma(\chi_{I_j} f) \in L^q$.

By reparametrization, we may assume that $b_j = 0$ and that $I_j \subset [0,\infty)$. Let $a_j$ be the left-hand endpoint of $I_j$. Applying an affine transformation if necessary, we may assume that $A_j = B_j = 1$ and $\gamma(a_j) = 0$.

Let $n_j = \lceil \log_2 a_j \rceil + 1$. To prove (3.3), it suffices by the triangle inequality to prove that
\[
\|\mathcal{E}_\gamma(\chi_{I_j \cap [\gamma a_j, \gamma \infty)} f)\|_{L^q(\mathbb{R}^d)} \lesssim \left\| \left( \sum_{n \geq n_j} \left| \mathcal{E}_\gamma(\chi_{I_{j,n}} f) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^d)}.
\]
(The left side excludes at most two of the $I_{j,n}$.) By the fundamental theorem of calculus,
\[
\gamma_1(t) = \int_{a_j}^{t} \gamma'_1(s) \, ds \sim t^j + 1 - a_j^j + 1 \sim t^j + 1
\]
on $I_j \cap [\gamma a_j, \gamma \infty)$. Thus for $n \geq n_j$, $\mathcal{E}_\gamma(\chi_{I_{j,n}} f)$ has its frequency support contained in
\[
\{ \xi \in \mathbb{R}^d : 2^n t^j \sim |\xi_1| \},
\]
and the proposition follows from by Fubini (to separate out the integral in $x_1$) and standard estimates for the Littlewood–Paley square function (see for instance [28, Ch. VI]).

\[\Box\]

**4. Almost orthogonality**

In this section we use a multilinear estimate to show that pieces at different scales interact weakly, and this allows us to sum. A key step in the multilinear estimate builds on an argument of Christ in [12], though we must interpolate to arrive at a useful estimate. The procedure for putting the pieces together is inspired by the bilinear approach to Fourier restriction from [32]. Bilinear and multilinear techniques have recently had considerable success in bounding Fourier restriction operators: [6, 21, 31, 33]. Additional motivation comes from some recent work [14, 15] on generalized Radon transforms.

**Proof of Theorem 1.1.** By the triangle inequality, it suffices to prove that
\[
\|\mathcal{E}_\gamma f\|_{L^q(\mathbb{R}^d)} \leq C_{N,d,p} \|f\|_{L^p(\lambda_\gamma)},
\]
when $f$ is supported on one of the intervals $I := I_j$ from the decomposition in Lemma 3.1. By Theorem 2.1, we already know that this estimate holds if $|L_\gamma| \sim A_j$ on $I_j$, so we may assume that $|L_\gamma(t)| \sim A_j |t - b_j|^{|k_j|}$ with $k_j \geq 1$. Making an affine transformation and reparametrization to $\gamma$ if necessary, we may further assume that $A_j = 1$, $b_j = 0$, and $\gamma(0) = 0$. As mentioned in the introduction, the theorem is known when $q \geq \frac{d^2 + 2d}{2}$, so we may assume that $q \leq d^2 + d$. These assumptions will remain in place for the remainder of the article.
To avoid unreadable subscripts, let $D = \frac{d + d}{2}$.

Let $I_n := I_{j,n}$ be the intervals from Proposition 3.2. By Proposition 3.2, arithmetic, Minkowski’s inequality (since $\frac{q}{p D} \leq 1$), and more arithmetic, we have

$$\|\mathcal{E}_\gamma f\|_q^q \lesssim \left( \sum_n \|\mathcal{E}_\gamma (\chi_{I_n} f)\|^2 \right)^{\frac{q}{2}} = \int \prod_{j=1}^D \left( \sum_{n_j} \|\mathcal{E}_\gamma (\chi_{I_{j,n_j}} f)\|^2 \right)^{\frac{q}{2p}} \, dx$$

$$\leq \int \prod_{j=1}^D \sum_{n_j} \|\mathcal{E}_\gamma (\chi_{I_{j,n_j}} f)\|^{\frac{q}{p}} \, dx \sim \sum_{n_1 \leq \cdots \leq n_D} \int \prod_{j=1}^D \|\mathcal{E}_\gamma (\chi_{I_{j,n_j}} f)\|^{\frac{q}{p}} \, dx.$$

**Lemma 4.1.** There exists $\varepsilon = \varepsilon_{d,N,p} > 0$ such that under the above assumptions on $I$, $\gamma$, and $q$, if $q = Dp'$, $n_1 \leq \cdots \leq n_D$, and $f_j$ is an $L^p$ function supported in $I_{n_j}$, $1 \leq j \leq D$, we have

$$\|\prod_{j=1}^D \mathcal{E}_\gamma f_j\|_{L^\frac{q}{D}} \leq C_{d,N,p} \varepsilon^2 \prod_{j=1}^D \|f_j\|_{L^p(\gamma_j)}.$$

(4.1)

We postpone the proof of the lemma for now to complete the proof of the theorem.

By the lemma and the estimate immediately preceding it,

$$\|\mathcal{E}_\gamma f\|_q^q \lesssim \sum_{n_1 \leq \cdots \leq n_D} 2^{-\varepsilon |n_D - n_1|} \prod_{j=1}^D \|\chi_{I_n} f\|^{q/D}_{L^p(\gamma_j)}.$$

(4.2)

Fix $n_1, n_D$. We recycle notation by defining

$$I_{n_1,n_D} := [2^{n_1}, 2^{n_D}].$$

Of course, if $n_1 \leq n_2 \leq \cdots \leq n_D$, then $I_{n_j} \subset I_{n_1,n_D}$; moreover, there are only $O((n_D - n_1)^D)$ choices for $n_2, \ldots, n_{D-1}$. Combining this with (4.2),

$$\|\mathcal{E}_\gamma f\|_q^q \lesssim \sum_{m \geq m} \sum_n 2^{-\varepsilon m} m^{C_d} \|\chi_{I_{n,n+m}} f\|^{q}_{L^p(\gamma_j)}.$$

(4.3)

Fix $m$. By Hölder’s inequality and the fact that each point is contained in at most $m$ intervals of the form $I_{n,n+m},$

$$\sum_n \|\chi_{I_{n,n+m}} f\|^{q}_{L^p(\gamma_j)} \leq \left( \sup_n \|\chi_{I_{n,n+m}} f\|^{q-p}_{L^p(\gamma_j)} \right) \sum_n \|\chi_{I_{n,n+m}} f\|^{p}_{L^p(\gamma_j)} \lesssim \sum_n \|f\|^{q}_{L^p(\gamma_j)}.$$

Combining this with (4.3),

$$\|\mathcal{E}_\gamma f\|_q^q \lesssim \sum_{m \geq m} 2^{-\varepsilon m} m^{C_d+1} \|f\|^{q}_{L^p(\gamma_j)} \lesssim \|f\|^{q}_{L^p(\gamma_j)}.$$

Thus the only thing left to establish Theorem 1.1 is the proof of Lemma 4.1. □

**Proof of Lemma 4.1.** By Hölder’s inequality,

$$\|\prod_{j=1}^D \mathcal{E}_\gamma f_j\|_{L^\frac{q}{D}} \leq \prod_{i=d+1}^D \|\mathcal{E}_\gamma f_{j_i}\|_{L^q} \prod_{i=1}^d \|\mathcal{E}_\gamma f_{j_i}\|_{L^\frac{q}{D}}.$$


wherever the $j_i$ are any enumeration of $\{1, \ldots, D\}$. Thus it suffices to prove that
\[
\| \prod_{j=1}^d E_j f_j \|_{L^{\frac{n}{2}}} \leq C_{N,d} 2^{-\varepsilon(n_d - n_1)} \prod_{j=1}^d \| f_j \|_{L^p(\lambda_\gamma)}, \text{ whenever } n_1 \leq \cdots \leq n_d. \quad (4.4)
\]

Next, by Hölder’s inequality and Theorem 2.1 (by our assumption on the supports of the $f_j$),
\[
\| \prod_{j=1}^d E_j f_j \|_{L^n} \leq \prod_{j=1}^d \| E_j f_j \|_{L^p} \leq C_{d,N,p} \prod_{j=1}^d \| f_j \|_{L^p(\lambda_\gamma)}.
\]

Thus it suffices to prove (4.4) when $n_d \geq n_1 + 2d$. Furthermore, by complex interpolation (with some $(p, q)$ sufficiently near the endpoint), it suffices to prove (4.4) when $q = d(d + 1)$ and $p = 2$. In other words, we have reduced matters to proving
\[
\| \prod_{j=1}^d E_j f_j \|_{L^{d+1}} \lesssim 2^{-\varepsilon(n_d - n_1)} \prod_{j=1}^d \| f_j \|_{L^2(\lambda_\gamma)}. \quad (4.5)
\]

By Hausdorff–Young,
\[
\| \prod_{j=1}^d E_j f_j \|_{L^{d+1}} \leq \|(d\mu_1) \ast \cdots \ast (d\mu_d)\|_{L^{\frac{d+1}{d+1}}}, \quad (4.6)
\]

where $d\mu_j$ is the measure defined by
\[
d\mu_j(\phi) = \int_{I_{n_j}} \phi(\gamma(t))^j f_j(t)\lambda(\gamma(t)) dt.
\]

We compute
\[
[(d\mu_1) \ast \cdots \ast (d\mu_d)](\phi) = \int \phi \left( \sum_{i=1}^d \gamma(t_i) \right) \prod_{i=1}^d f_i(t_i)\lambda(t_i) dt = \sum_{\sigma \in S_d} \int_{P_{\sigma}} \phi \left( \sum_{i=1}^d \gamma(t_i) \right) \prod_{i=1}^d f_i(t_i)\lambda(t_i) dt,
\]

where $S_d$ is the symmetric group on $d$ letters, and for each $\sigma \in S_d$,
\[
P_{\sigma} = \{(t_1, \ldots, t_d) \in I_{n_1} \times \cdots \times I_{n_d} : t_{\sigma(1)} < \cdots < t_{\sigma(d)}\}.
\]

By Lemma 3.1, $\Phi(t) := \sum_{j=1}^d \gamma(t_j)$ is one-to-one on $P_{\sigma}$, so we make the change of variables $\xi = \Phi(t)$, yielding
\[
[(d\mu_1) \ast \cdots \ast (d\mu_d)] = \sum_{\sigma \in S_d} F_{\sigma},
\]

where
\[
F_{\sigma}(\xi) = \chi_{P_{\sigma}}(t) \prod_{i=1}^d f_i(t_i)\lambda(t_i)|J_{\gamma}(t_1, \ldots, t_d)|^{-1}|t|^{-1}|(\Phi_{\sigma})^{-1}(\xi)|.
\]

By the change of variables formula and the geometric inequality (3.2),
\[
\| F_{\sigma} \|_{L^{\frac{d+1}{d+1}}} = \| \chi_{P_{\sigma}}(t) \prod_{i=1}^d f_i(t_i)\lambda(t_i)|J_{\gamma}(t_1, \ldots, t_d)|^{-1}|t|^{-1\frac{1}{d+1}} \|_{L^{\frac{d+1}{d+1}}}
\]
By the pigeonhole principle, there exists an index \( k, 1 \leq k < d \) such that \( n_{k+1} - n_k \geq \frac{n_2 - n_1}{d} \). In particular, \( n_{k+1} - n_k \geq 2 \). Therefore for \((t_1, \ldots, t_d) \in I_{n_1} \times \cdots \times I_{n_d} \),

\[
\prod_{1 \leq i < j \leq d} |t_i - t_j| = \prod_{i \leq k, j \geq k+1} |t_i - t_j| \prod_{1 \leq i < j \leq k} |t_i - t_j| \prod_{k+1 \leq i < j \leq d} |t_i - t_j| \sim \prod_{i \leq k, j \geq k+1} 2^{n_j} \prod_{1 \leq i < j \leq k} |t_i - t_j| \prod_{k+1 \leq i < j \leq d} |t_i - t_j| .
\]

This implies that

\[
\|F_\sigma \|_{L^{d+k+1}_2} \sim 2^{-\frac{1}{2}(n_{k+1} + \cdots + n_d)} T_k(f_1, \ldots, f_k) \times T_{d-k}(f_{k+1}, \ldots, f_d),
\]

where

\[
T_\ell(g_1, \ldots, g_\ell) = \int_{\mathbb{R}^\ell} \prod_{i=1}^{\ell} \left( \prod_{1 \leq i < j \leq \ell} g_{i,j}(t_i - t_j) \right) dt \lesssim \prod_{i=1}^{\ell} \|f_i\|_{L^p} \prod_{1 \leq i < j \leq \ell} \|g_{ij}\|_{L^{q, \infty}}.
\]

We apply this to \( T_k \) with \( q = d \) and \( p = \frac{2d}{2d-k+1} \) (if \( k = 1, p = k \), but then the inequality is trivial) to see that

\[
|T_k(f_1, \ldots, f_k)| \lesssim \prod_{i=1}^{k} \|f_i\|_{L^p} \|f_i\|^\frac{1}{2} \; L^{d+k+1}(\lambda_i).
\]

Since \( \frac{2(d+1)}{2d-k+1} < 2 \), by Hölder’s inequality and the fact that \( |\text{supp} \, f_j| \leq |I_{n_j}| \approx 2^{n_j} \), this implies that

\[
|T_k(f_1, \ldots, f_k)| \lesssim \prod_{i=1}^{k} \|f_i\|_{L^2(\lambda_i)}^{\frac{2n_i}{d} - \frac{1}{d}}.
\]

Similarly,

\[
|T_{d-k}(f_1, \ldots, f_k)| \lesssim \prod_{i=k+1}^{d} \|f_i\|_{L^2(\lambda_i)}^{\frac{2n_i}{d} - \frac{1}{d}}.
\]

Inserting these estimates into (4.7) and performing a bit of arithmetic,

\[
\|F_\sigma\|_{L^{d+k+1}_2} \lesssim 2^{\frac{(n_1 + \cdots + n_k)(d-k)}{d} - \frac{(n_{k+1} + \cdots + n_d)k}{d}} \prod_{i=1}^{d} \|f_i\|_{L^2(\lambda_i)}
\]

\[
\lesssim 2^{-\frac{k(d-k)}{d} \frac{(n_k - n_{k+1})}{d}} \prod_{i=1}^{d} \|f_i\|_{L^2(\lambda_i)}.
\]

Since \( n_k - n_{k+1} \geq \frac{1}{d}(n_d - n_1) \) by our choice of \( k \), this completes the proof of (4.4) and hence of Lemma 4.1. \( \square \)
5. Proof of the corollary

We begin with (1.4). Let \( I_{lo} = \{|L_\gamma| < 1\} \) and \( I_{hi} = \mathbb{R} \setminus I_{lo} \). With \( Z_\gamma \) equal to the set of complex zeroes, we estimate

\[
1 \gtrsim |L_\gamma| \gtrsim d_\gamma^{K_{\max}}, \quad \text{on } I_{lo}, \quad |L_\gamma| \sim d_\gamma^{K_{\max}} \gtrsim 1, \quad \text{on } I_{hi}. \tag{5.1}
\]

If \( p \leq q \), the full range of estimates follow by writing \( \hat{f} \circ \gamma = (\hat{f} \circ \gamma) \chi_{I_{lo}} + (\hat{f} \circ \gamma) \chi_{I_{hi}} \) and applying (5.1), (1.3), and the embedding \( L^q \subseteq L^p \).

Now assume that \( p < q \) and \( N_{\min q} < p' < N_{\max q} \). Let \( I_n = \{2^n \leq |L_\gamma| < 2^{n+1}\} \). Then

\[
|I_n| \sim 2^{n/K_{\min}}, \quad n \leq 0, \quad |I_n| \sim 2^{n/K_{\max}}, \quad n \geq 0. \tag{5.2}
\]

(It is crucial that \( \gamma \) is a polynomial.) Let \( \hat{q} = \frac{2}{2+q}p' \); then \( q < \hat{q} \). By Hölder’s inequality, (5.2), and our main theorem, if \( n \geq 0 \),

\[
\|\hat{f} \circ \gamma\|_{L^q(I_n)} \lesssim 2^{n\left(\frac{1}{K_{\max}} - \frac{1}{K_{\max}} - \frac{2}{(2+q)p'}\right)} \|\hat{f} \circ \gamma\|_{L^{p'}(\mathbb{R}^d)}.
\]

The exponent is just \( \frac{1}{K_{\max}}(\frac{1}{q} - N_{\max q}) < 0 \), so this is summable over \( n \geq 0 \). The argument when \( n < 0 \) is similar.

Now we turn to the optimality of (1.4). The condition \( p < p_d = \frac{d^2+4d+2}{d^2+2} \) for \( q \geq 1 \) follows by considering a nondegenerate segment along \( \gamma \) and applying a result of Arkhipov–Chubarikov–Kuratsuba in [1] (see also [7]). That \( p < p_d \) for \( q < 1 \) follows by interpolation with the restricted strong type \( L^{p_d,1} \rightarrow L^{p_d} \) estimate for nondegenerate curves from [2]. It suffices by interpolation to verify the remaining inequalities when \( q \geq 1 \). Thus we may consider \( L^{q'} \rightarrow L^{p'} \) bounds for the extension operator. The condition \( N_{\min q} \leq p' \leq N_{\max q} \) follows from Knapp type examples. Assume \( p > q \). Performing an affine transformation and reparametrization, we may assume that \( \gamma(t) = (t^{m_1}\theta_1(t), \ldots, t^{m_d}\theta_d(t)) \), where \( m_1 + \cdots + m_d = N_{\min} \) and the \( \theta_i \) are polynomials with \( \theta_i(0) = 1 \) and \( \|\theta'_i\|_{C^0} \) sufficiently small on \([0,1]\). Let \( N \) be a large integer and define

\[
g_n(t) = 2^\frac{n}{p} e^{ix_n\gamma(t)}\chi_{[2^{-n},2^{-n+1})}, \quad 1 \leq n \leq N; \quad g = \sum_{n=1}^N g_n,
\]

where the \( x_n \) are spaced sufficiently far apart in \( \mathbb{R}^d \). Using Knapp-type arguments and the physical space separation of the \( \mathcal{F}_{\gamma}g_n \), if \( p' = N_{\min q} \),

\[
\|g\|_{L^{p'}} \sim N^{\frac{1}{p'}}, \quad \|\mathcal{F}_{\gamma}g\|_{L^{p'}} \sim N^{\frac{1}{p'}}.
\]

Letting \( N \to \infty \), since \( q' > p' \), \( L^{q'} \rightarrow L^{p'} \) boundedness cannot hold. The verification of \( p' < N_{\max q} \) is similar.

This leaves us to prove (1.3). We begin with a well-known lemma.

Lemma 5.1. Let \( P \) be a real polynomial and \( Z_\gamma \) be the set of complex zeroes of \( P \). There exists a decomposition \( \mathbb{R} = \bigcup_{j=1}^{\deg P} I_j \) as a union of intervals such that the following holds for each \( j \). There exist \( C_j > 0 \), \( b_j \in Z_\gamma \), and \( k_j \in \mathbb{Z}_{\geq 0} \) so that for every \( t \in I_j \), \( b_j \) is the closest element of \( Z_\gamma \) to \( t \) and \( |P(t)| \sim C_j|t-b_j|^{k_j} \). The implicit constant depends only on \( \deg P \).

This is a simple variant of a well known (see e.g. [10]) lemma; for the convenience of the reader, we give the short proof.
Proof. Define $I_b = \{ t \in \mathbb{R} : d_\gamma = |t - b| \}$, $b \in Z_\gamma$. Of course, $\mathbb{R} = \bigcup_b I_b$, so we may leave $b$ fixed for the remainder of the argument. We index the elements of $Z_\gamma$ by $b = b_0, \ldots, b_M$, so that $|b - b_0| \leq \cdots \leq |b - b_M|$. For convenience, we also set $b_{M+1} = \infty$. Define a sequence of annuli,

$$A_j = \{ t \in I_b : \frac{1}{2}|b - b_j| \leq |t - b| \leq \frac{1}{2}|b - b_{j+1}| \}, \quad 0 \leq j \leq M.$$ 

Then $I_b = \bigcup A_j$, and on $A_j$, by the triangle inequality and the definition of $I_b$,

$$|P(t)| = C \prod_{k=0}^M |t - b_k|^{n_k} \sim C \prod_{k=j+1}^M |b - b_k|^{n_k} |t - b|^{n_0 + \cdots + n_j},$$

where we are taking $n_k = 0$ if $P(b_k) \neq 0$.

Now we return to the proof of (1.3). We adapt an argument of Drury–Marshall [18] to the polynomial case. By the triangle inequality, it suffices to prove the estimate when the $L_1^{q_+}$ norm is restricted to one of the intervals $I = I_j$ from our lemma. Let $C, k, b$ denote the corresponding constant, power, and zero. In particular, $d_r(t) = |t - b|$ on $I$. Applying an affine transformation to $\gamma$ if necessary (this leaves (1.3) invariant), we may assume that $C = 1$.

We rewrite our main result (1.1) as

$$\|\hat{f}(\gamma(t))|t - b|^{\frac{-1}{2}}\|_{L^{r}(\lambda_r, \gamma(t) dt)} \leq C_{p, d, N} \|f\|_{L^p_r}, \quad p' = \frac{d(d+1)}{2} q, \quad 1 \leq p < \frac{d^2 + d + 2}{d^2 + d}.$$ 

For any $r > 0$, $|t - b|^{-\frac{1}{2}}$ is in $L^{r, \infty}$, so by the Lorentz space version of Hölder’s inequality [29],

$$\|\hat{f}(\gamma(t))|t - b|^{\frac{-1}{2} + \frac{1}{q}}\|_{L^{r,q}_\gamma} \leq C_{p, c, d, N} \|f\|_{L^p_r},$$

whenever $p' = \frac{d(d+1)}{2} q, \ c \leq q, \ 1 \leq p < \frac{d^2 + d + 2}{d^2 + d}$.

Marcinkiewicz interpolation [29] along the level sets of $\frac{\lambda_r}{p'} - \frac{1}{c} + \frac{1}{q'}$, within the region $p' = \frac{d(d+1)}{2} q, \ c \leq q, \ 1 < p < \frac{d^2 + d + 2}{d^2 + d}$, gives the estimate

$$\|\hat{f}(\gamma(t))|L_2(t)|^{\frac{-1}{2} + \frac{d(d+1)}{2p'} d_r(t)}\|_{L^{p_r}(I)} \leq C_{p, q, N, k} \|f\|_{L^p_r},$$

and since $0 \leq k \leq N$, we can put the pieces back together to obtain (1.3).

References


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