Fix field \( F \)

Given \( F \)-vector spaces \( V, W \) over \( F \)

Given

\[
\text{basis } \{e_i\}_{i=1}^m \text{ for } V
\]

\[
\text{basis } \{f_i\}_{i=1}^n \text{ for } W
\]

For \( \psi \in \text{Hom}_F(V, W) \), consider

matrix \( A \) rep \( \psi \) wrt \( X \) and \( Y \)

\[
\psi(v) = \sum_{i=1}^m A_{ij} v_i
\]

So

\( A \in \text{Mat}_{m \times n}(F) \).

The map

\[
\text{Hom}_F(V, W) \rightarrow \text{Mat}_{m \times n}(F)
\]

\( \psi \rightarrow \text{matrix rep } \psi \) wrt \( X \) and \( Y \)

is isom of vector spaces.
Cor 3. Given finite dim'd vector spaces $V, W$ over $F$, then the dimension of $\text{Hom}_F(V, W)$ is $\dim(V) \times \dim(W)$.

pf. By Prop 2. \qed
Given three finite-dim'l vector spaces over $F$:

$U$, $V$, $W$.

Given

- basis $\{ x_1, x_2, \ldots \}$ for $U$
- basis $\{ v_1, v_2, \ldots \}$ for $V$
- basis $\{ w_1, w_2, \ldots \}$ for $W$

Given linear transformations:

$U \rightarrow V \rightarrow W$

$\phi$ $\phi$

Consider

- matrix rep $\phi$ rel $\phi$ is $\star$, $\star$ $\star$
- matrix rep $\phi \circ \phi$ rel $\phi \circ \phi$ is $\star$, $\star$ $\star$ $\star$

- matrix rep $\phi \circ \phi \circ \phi$ rel $\phi \circ \phi \circ \phi$ is $\star$, $\star$ $\star$ $\star$

$A = \star$, $\star$, $\star$ $\star$

$B = \star$, $\star$ $\star$ $\star$

$C = \star$, $\star$ $\star$ $\star$

How are $A$, $B$, and $C$ related?
For \( 1 \leq j \leq r \)

\[
(\phi \circ \psi)(w) = \sum_{i=1}^{t} C_{ij} w_i
\]

Also

\[
(\phi \circ \psi)(u_x) = \phi(\psi(u_x))
\]

\[
= \phi\left( \sum_{l=1}^{a} A_{lx} \psi(v_l) \right)
\]

\[
= \sum_{l=1}^{a} A_{lx} \phi(v_l)
\]

\[
= \sum_{l=1}^{a} A_{lx} \left( \sum_{i=1}^{t} B_{il} \psi(w_i) \right)
\]

\[
= \sum_{i=1}^{t} \left( \sum_{l=1}^{a} B_{il} A_{lx} \right) w_i
\]

Find is 1st component \( w_i \) - coefs to get

\[
C_{ij} = \sum_{l=1}^{a} B_{il} A_{lx}
\]

In other words

\[
C = \beta A
\]

\[\text{matrix product}\]
Cor 4. The matrix product is associative.

pf Given matrices $A, B, C$

show

$$(AB)C = A(BC)$$

[assume the dimensions are such that above products make sense]

View $A, B, C$ as representing some linear maps $\varphi, \phi, \psi$ w.r.t. some given bases.

$AB$ represents $\varphi \circ \phi$

$$(AB)C$$ represents $$(\varphi \circ \phi) \circ \psi$$

$BC$ represents $\phi \circ \psi$

$A(BC)$ represents $\varphi (\phi \circ \psi)$

But

$$(\varphi \circ \phi) \circ \psi = \varphi (\phi \circ \psi)$$

so * holds.
For $n \geq 1$ define

$$\text{Mat}_n(F) = \text{Mat}_{n \times n}(F)$$

Matrix multiplication turns $\text{Mat}_n(F)$ into a ring with identity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The map

$$F \to \text{Mat}_n(F)$$

$$\gamma : \quad a \to aI$$

turns $\text{Mat}_n(F)$ into an $F$-algebra.

Given vector space $V$ over $F$ with $\dim(V) = n$.

Recall $F$-only

$$\text{End}_F(V) = \text{Hom}_F(V,V)$$

Fix a basis $\{v_i : i = 1, \ldots, n\}$ for $V$.

For $\gamma \in \text{End}_F(V)$, there is a unique $A \in \text{Mat}_n(F)$ such that

$$\gamma(v_i) = \sum_{i=1}^n A_{ij} v_j$$

Call $A$ the matrix that represents $\gamma$ with respect to $\ast$

Note if $1 \in \text{End}_F(V)$

$$I = \text{matrix that reps } 1 \text{ rel } \ast.$$
Prop 5. With above notation, the map

\[ \text{End}_F(V) \rightarrow \text{Mat}_n(F) \]

\[ \phi \rightarrow \text{matrix rep } \phi \text{ rel } \ast \]

is an \( F \)-algebra isomorphism.

Proof. \( \phi \) is vector space iso by Prop 2 (with \( V = W \)).

Observe \( \phi = I \).

Show \( \phi \) respects mult.

For \( \phi, \psi \in \text{End}_F(V) \),

\[ (\phi \circ \psi) \ast = \phi \ast \psi \ast \]

This is just \( c = \text{BA} \) from above for \( \ast \).

\( \square \)
Transition matrices

Given a vector space $V$ over $F$

Given

- basis $\{e_1, \ldots, e_n\}$ for $V$
- basis $\{e'_1, \ldots, e'_n\}$ for $V$

There is a unique matrix $S \in \text{Mat}_n(F)$ such that

$$ v'_j = \sum_{i=1}^n S_{ij} v_i $$

Call $S$ the transition matrix from $V$ to $V'$

Given

- basis $\{e_1, \ldots, e_n\}$ for $V$

Compare

- transition matrix from $V$ to $V'$ $(= R)$
- transition matrix from $V$ to $V''$ $(= S)$
- transition matrix from $V'$ to $V''$ $(= T)$
For all \( n \),

\[
W_1 = \sum_{i=1}^{n} R_{i1} u_i \tag{1}
\]

Also,

\[
W_1 = \sum_{\lambda=1}^{\hat{n}} T_{\lambda} u_1
\]

\[
= \sum_{\lambda=1}^{\hat{n}} T_{\lambda} \left( \sum_{i=1}^{n} S_{i\lambda} u_i \right)
\]

\[
= \sum_{i=1}^{n} \left( \sum_{\lambda=1}^{\hat{n}} S_{i\lambda} T_{\lambda} \right) u_i
\]

For all \( n \), compare \( u_i \) - coeff to get

\[
R_{ij} = \sum_{\lambda=1}^{\hat{n}} S_{i\lambda} T_{\lambda j} \quad 1 \leq i, j \leq n
\]

In other words

\[
R = ST
\]
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Given a vector space $V$ over $F$

Given basis $\{u_1, \ldots, u_n\}$ for $V$

Let $S$ be transormation from $V$ to $\mathbb{R}^n$.

Then $S$ is invertible in $\text{Mat}_n(F)$

Moreover $S^{-1}$ is the transpose matrix for $\mathbb{R}^n$ to $V$.

pf \hspace{1cm} \text{In the discussion above the lemma}

take $w_i = u_i$ for $i = 1, \ldots, n$

Then $x = x^T$ so

$I = R$

$= S T$

So $T = S^{-1}$

$\square$
Given two vector spaces $V$, $W$

Given

- basis $\{v_1, \ldots, v_n\}$ for $V$
- basis $\{v_1, \ldots, v_n\}$ for $V'$
- basis $\{w_1, \ldots, w_m\}$ for $W$
- basis $\{w_1, \ldots, w_m\}$ for $W'$

Given linear transformation $\phi : V \rightarrow W$

Compare

- matrix map $\phi$ with $x$ and $x'$
- matrix map $\phi'$ with $x'$ and $x''$

Let

$S = \text{trans matrix for } x \rightarrow x'$

$T = \text{trans matrix for } x' \rightarrow x''$
\[ F_a \triangleq \psi(v) = \sum_{i=1}^{m} A_{i} v_i \]

\[ \psi(v') = \sum_{a=1}^{\infty} A_{a} v'_a \]

\[ \psi \left( \sum_{r=1}^{n} S_{r} v_r \right) = \sum_{a=1}^{\infty} A_{a} \left( \sum_{i=1}^{m} T_{i,a} v_i \right) \]

\[ \sum_{r=1}^{n} S_{r} \psi(v) = \sum_{a=1}^{\infty} \left( \sum_{i=1}^{m} T_{i,a} A_{a} \right) v_i \]

\[ \sum_{r=1}^{n} S_{r} \left( \sum_{i=1}^{m} A_{i} v_i \right) = \sum_{a=1}^{\infty} \left( \sum_{r=1}^{n} A_{a} S_{r} \right) v_i \]

\[ \sum_{r=1}^{n} A_{i_r} S_{r} = \sum_{a=1}^{\infty} T_{a,i} A_{a} \]  

So

In other words

\[ A S = T A' \]

or

\[ A' = T^{-1} A S \]