30. We proceed in stages:

<table>
<thead>
<tr>
<th>stage</th>
<th>to do</th>
<th># choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>pick gender to the parent’s right</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>order the girls clockwise</td>
<td>5!</td>
</tr>
<tr>
<td>3</td>
<td>order the boys clockwise</td>
<td>5!</td>
</tr>
</tbody>
</table>

The answer is $2 \times (5!)^2$.

Now assume that there are two parents, labelled P and Q. Suppose we move clockwise around the table from P to Q. Let $n$ denote the number of seats between the two. Thus $n = 0$ (resp. $n = 10$) if Q sits next to P at P’s left (resp. right). We now partition the set of solutions according to the value of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th># seatings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2(5!)^2$</td>
</tr>
<tr>
<td>1</td>
<td>$2(5!)^2$</td>
</tr>
<tr>
<td>2</td>
<td>$4(5!)^2$</td>
</tr>
<tr>
<td>3</td>
<td>$2(5!)^2$</td>
</tr>
<tr>
<td>4</td>
<td>$4(5!)^2$</td>
</tr>
<tr>
<td>5</td>
<td>$2(5!)^2$</td>
</tr>
<tr>
<td>6</td>
<td>$4(5!)^2$</td>
</tr>
<tr>
<td>7</td>
<td>$2(5!)^2$</td>
</tr>
<tr>
<td>8</td>
<td>$4(5!)^2$</td>
</tr>
<tr>
<td>9</td>
<td>$2(5!)^2$</td>
</tr>
<tr>
<td>10</td>
<td>$2(5!)^2$</td>
</tr>
</tbody>
</table>

The answer is the sum of the entries in the right-most column, which comes to $30 \times (5!)^2$.

38. We make a change of variables. Define

$$y_1 = x_1 - 2, \quad y_2 = x_2, \quad y_3 = x_3 + 5, \quad y_4 = x_4 - 8.$$ 

Note that $\{x_i\}_{i=1}^4$ is a solution to the original problem if and only if

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0, \quad y_4 \geq 0, \quad y_1 + y_2 + y_3 + y_4 = 25.$$ 

Therefore the number of solutions is

$$\binom{25 + 4 - 1}{4 - 1} = \binom{28}{3}.$$ 

39. (a) There are $\binom{20}{6}$ ways to choose six sticks from the twenty available sticks.
(b) Label the sticks 1, 2, \ldots, 20. Suppose we choose six sticks labelled \( \{x_i\}_{i=1}^{6} \) with \( x_1 < x_2 < \cdots < x_6 \). Define

\[
\begin{align*}
y_1 &= x_1 - 1, \\
y_2 &= x_2 - x_1 - 2, \\
y_3 &= x_3 - x_2 - 2, \\
y_4 &= x_4 - x_3 - 2, \\
y_5 &= x_5 - x_4 - 2, \\
y_6 &= x_6 - x_5 - 2, \\
y_7 &= 20 - x_6.
\end{align*}
\]

Observe that the solutions \( \{x_i\}_{i=1}^{6} \) to the original problem correspond to the integral solutions \( \{y_i\}_{i=1}^{7} \) for

\[
y_i \geq 0 \quad (1 \leq i \leq 7), \quad \sum_{i=1}^{7} y_i = 9.
\]

Therefore the number of solutions \( \{x_i\}_{i=1}^{6} \) to the original problem is

\[
\binom{9 + 7 - 1}{7 - 1} = \binom{15}{6}.
\]

(c) We proceed as in (b) with the modification

\[
\begin{align*}
y_1 &= x_1 - 1, \\
y_2 &= x_2 - x_1 - 3, \\
y_3 &= x_3 - x_2 - 3, \\
y_4 &= x_4 - x_3 - 3, \\
y_5 &= x_5 - x_4 - 3, \\
y_6 &= x_6 - x_5 - 3, \\
y_7 &= 20 - x_6.
\end{align*}
\]

The solutions \( \{x_i\}_{i=1}^{6} \) to the original problem correspond to the integral solutions \( \{y_i\}_{i=1}^{7} \) for

\[
y_i \geq 0 \quad (1 \leq i \leq 7), \quad \sum_{i=1}^{7} y_i = 4.
\]

Therefore the number of solutions \( \{x_i\}_{i=1}^{6} \) to the original problem is

\[
\binom{4 + 7 - 1}{7 - 1} = \binom{10}{6}.
\]

41. We proceed in stages:

<table>
<thead>
<tr>
<th>stage</th>
<th>to do</th>
<th># choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>hand out the orange</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>give one apple to each of the other children</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>distribute remaining 10 apples to 3 children</td>
<td>( \binom{12}{2} )</td>
</tr>
</tbody>
</table>

The answer is \( 3 \times \binom{12}{2} \).

42. We proceed in stages:

<table>
<thead>
<tr>
<th>stage</th>
<th>to do</th>
<th># choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>hand out lemon drink</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>hand out lime drink</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>give one orange drink to each of the remaining two students</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>distribute remaining 8 orange drinks to 4 students</td>
<td>( \binom{11}{3} )</td>
</tr>
</tbody>
</table>
The answer is the product of the entries in the right-most column, which comes to $12 \times \left( \frac{11}{3} \right)$.

45. (a) For $1 \leq i \leq 5$ let $x_i$ denote the number of books on shelf $i$. We seek the number of integral solutions to

$$x_i \geq 0 \quad (1 \leq i \leq 5), \quad \sum_{i=1}^{5} x_i = 20.$$ 

The answer is

$$\binom{20 + 5 - 1}{5 - 1} = \binom{24}{4}.$$ 

(b) We proceed in stages:

<table>
<thead>
<tr>
<th>stage</th>
<th>to do</th>
<th># choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>put book 1 on a shelf</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>put book 2 on a shelf</td>
<td>5</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>20</td>
<td>put book 20 on a shelf</td>
<td>5</td>
</tr>
</tbody>
</table>

The answer is the product of the entries in the right-most column, which comes to $5^{20}$.

(c) We proceed in stages:

<table>
<thead>
<tr>
<th>stage</th>
<th>to do</th>
<th># choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>order the books</td>
<td>$20!$</td>
</tr>
<tr>
<td>2</td>
<td>pick a solution ${x_i}_{i=1}^5$ to part (a)</td>
<td>$\binom{24}{4}$</td>
</tr>
<tr>
<td>3</td>
<td>put the first $x_1$ books on shelf 1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>put the next $x_2$ books on shelf 2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>put the next $x_3$ books on shelf 3</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>put the next $x_4$ books on shelf 4</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>put the last $x_5$ books on shelf 5</td>
<td>1</td>
</tr>
</tbody>
</table>

The answer is the product of the entries in the right-most column, which comes to $20! \times \binom{24}{4}$.

55. (a) The word has 17 letters with repetitions

<table>
<thead>
<tr>
<th>letter</th>
<th>A</th>
<th>B</th>
<th>D</th>
<th>E</th>
<th>H</th>
<th>I</th>
<th>K</th>
<th>O</th>
<th>P</th>
<th>R</th>
<th>S</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>mult</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The number of permutations is

3
\[ \frac{17!}{2! \times (3!)^2} \]

(b) The word has 29 letters with repetitions

\[
\begin{array}{c|cccccccccccc}
\text{letter} & A & C & F & H & I & L & N & O & P & T & U \\
\hline
\text{mult} & 2 & 4 & 2 & 9 & 3 & 3 & 2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

The number of permutations is

\[ \frac{29!}{(2!)^3 \times (3!)^2 \times 4! \times 9!} \]

(c) The word has 45 letters with repetitions

\[
\begin{array}{c|cccccccccccc}
\text{letter} & A & C & E & I & L & M & N & O & P & R & S & T & U & V \\
\hline
\text{mult} & 2 & 6 & 1 & 6 & 3 & 2 & 4 & 9 & 2 & 2 & 4 & 1 & 2 & 1 \\
\end{array}
\]

The number of permutations is

\[ \frac{45!}{(2!)^5 \times 3! \times (4!)^2 \times (6!)^2 \times 9!} \]

(d) The number of permutations is 15!.

60. The number of ways to pick the 15 bagels is equal to the number of integral solutions for

\[ x_i \geq 0 \quad (1 \leq i \leq 6), \quad \sum_{i=1}^{6} x_i = 15 \]

which comes to \( \binom{15+6-1}{6-1} = \binom{20}{5} \). This is the denominator. We now compute the numerator for the first probability. The number of ways to pick the 15 bagels so that you get at least one bagel of each kind is equal to the number of integral solutions for

\[ y_i \geq 0 \quad (1 \leq i \leq 6), \quad \sum_{i=1}^{6} y_i = 9 \]

which is \( \binom{9+6-1}{6-1} = \binom{14}{5} \). The first desired probability is

\[ \frac{\binom{14}{5}}{\binom{20}{5}} \].
We now compute the numerator for the second probability. The number of ways to pick the 15 bagels so that you get at least three sesame bagels is equal to the number of integral solutions for

\[ z_i \geq 0 \quad (1 \leq i \leq 6), \quad \sum_{i=1}^{6} z_i = 12 \]

which is \( \binom{12 + 6 - 1}{6 - 1} = \binom{17}{5} \). The second desired probability is

\[ \frac{\binom{17}{5}}{\binom{20}{5}}. \]

61. The sample space \( S \) satisfies

\[ |S| = 9! \times \binom{9}{4}. \]

The first event \( E \) satisfies

\[ |E| = 9!. \]

The first desired probability is \( |E|/|S| \) which comes to \( \binom{9}{4}^{-1} \). The second event \( F \) satisfies

\[ |F| = 5! \times 4!. \]

The second desired probability is \( |F|/|S| \) which comes to

\[ \frac{5! \times 4!}{9! \times \binom{9}{4}}. \]

63. The size of the sample space is \( 6^4 \). This is the denominator for (a)--(e) below.

(a) The die numbers must be some permutation of 3, 1, 1, 1 (4 ways) or 2, 2, 1, 1 (\( \binom{4}{2} \) ways). The numerator is \( 4 + 6 = 10 \). The desired probability is \( 10/6^4 \).

(b) One dot occurs either once (\( 4 \times 5^3 \) ways) or twice (\( \binom{4}{2} \times 5^2 \) ways) or not at all (\( 5^4 \) ways). The desired probability is

\[ \frac{5^4 + 4 \times 5^3 + \binom{4}{2} \times 5^2}{6^4}. \]

(c) The desired probability is \( 5^4/6^4 \).

(d) The desired probability is \( P(6,4)/6^4 \).

(e) The die numbers must be some permutation of \( i, i, j \) (\( 4 \times 6 \times 5 \) ways) or \( i, i, j \) (\( \binom{6}{2} \times \binom{4}{2} \) ways). Here we mean \( i \neq j \). The desired probability is

\[ \frac{4 \times 6 \times 5 + \binom{6}{2} \times \binom{4}{2}}{6^4}. \]