Lectures 25 - 26: Bulk Scaling of the General $\beta$-Ensemble 
(continued)

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1 Finishing the proof

Recall the definitions of the random $n$-vector $\Lambda_n$ and the semicircle density $\sigma$. We continue with our goal to prove that for any $c \in (-2, 2)$,

$$\sigma(c)\sqrt{n}(\Lambda_n - c\sqrt{n}) \Longrightarrow \text{Sine}_\beta,$$

where $\text{Sine}_\beta$ is a translation invariant point process which will be specified by its counting function $N(\lambda)$.

Let $Z$ be a two dimensional Brownian Motion and let $\alpha_\lambda$ be given by the stochastic differential equation

$$d\alpha_\lambda = \frac{\lambda\beta}{4} e^{-\frac{\beta}{4} t} dt + \text{Re} \left( (e^{-i\alpha_\lambda} - 1) \, dZ \right)$$

with $\alpha_\lambda(0) = 0$. We showed last time that $N(\lambda)$ is specified by the limit,

$$N(\lambda) = \frac{1}{2\pi} \lim_{t \to +\infty} \alpha_\lambda(t).$$

In the last lecture we formally derived the SDE (1.2), by relating the eigenvalue problem to a certain discrete Markov chain and then arguing that as $n \to \infty$, this discrete Markov chain converges to the continuous time Markov process described by the SDE (1.2). However there are several problems in making these arguments precise. Here we list the problems.

1. The time steps of the discrete Markov chain should be infinitesimal ($O\left(\frac{1}{N}\right)$). Recall the formula for $s_j$ from the last lecture. Observe that,

$$\frac{s_j}{\sqrt{n}} = \sqrt{1 - \frac{j}{n} - \frac{1}{2n}} \approx \sqrt{1 - t}$$

where $t \approx \frac{j}{n}$.

Note that $\sqrt{1 - t}$ has an unbounded derivative at $t = 1$. In fact,

$$\frac{s_{n-k}}{\sqrt{n}} = \sqrt{\frac{k}{n}}.$$

Hence the change is of order $1/\sqrt{n}$ for $k$ near $n$ and this is a problem.
2. The formal argument of last lecture only works when $c = 0$. We need to figure out what happens for other values of $c$. Note that we center the process by $c\sqrt{n}$ which gets big as $n$ gets large.

The actual derivation of the result has a lot more steps to take care of these issues and now we sketch the main steps. Let’s go back to the eigenvalue equation for the reduced matrix we derived last time.

\[ s_t u_l + X_i u_{l+1} + (Y_i + s_l) u_{l+2} = \lambda u_{l+1}. \]  

(1.4)

$u_0 = 1$, $u_1 = 1$ and let

\[ \Lambda = c\sqrt{n} + \frac{\lambda}{2\sigma(c)\sqrt{n}}. \]

We have \( t \approx \frac{l}{n} \) and \( \frac{s_l}{\sqrt{n}} \approx \sqrt{1 - t} \).

Divide equation (1.4) by $\sqrt{n}$ to get the “main part” of the evolution of $u$ as

\[ s(t) u_l + s(t) u_{l+2} = C u_{l+1}. \]

This is a three term recursion and can be written as

\[ u_{l+2} = c \frac{s}{s} u_{l+1} - u_l. \]

The characteristic equation of this recursion is

\[ q^2 - \frac{c}{s} q + 1 = 0. \]

The $q_1$ and $q_2$ be the roots of the above equation. Then $q_1 q_2 = 1$ and the general solution of the recursion is

\[ u_l = \alpha_1 q_1^l + \alpha_2 q_2^l = \alpha_1 q_1^l + \bar{\alpha}_1 q_1^{-l} = 2 \text{Re} \left( \alpha_1 q_1^l \right). \]

\[ q_{1,2} = \frac{\frac{c}{s} \pm \sqrt{(\frac{c}{s})^2 - 4}}{2} = \frac{c}{2s} \pm i \sqrt{1 - \left( \frac{c}{2s} \right)^2}. \]

We will work in the case where \( \left( \frac{c}{s} \right)^2 < 4 \).
and we have complex roots and thus bounded solutions. Note that

\[
\left( \frac{c}{s} \right)^2 < 4
\]

is equivalent to

\[
t < 1 - \left( \frac{c}{2} \right)^2.
\]

In the case when \( c = 0 \), \( q_1 = i \) and \( q_2 = -i \). In the time interval \([0, 1 - (\frac{c}{2})^2]\) the main part of the evolution of \( u \) is governed by \( O(1) \) steps, and hence it is not infinitesimal. We will try to eliminate the \( O(1) \) part and hope that the remaining part is infinitesimal.

As before set

\[
r_l = \frac{u_{l+1}}{u_l}.
\]

Then from the recursion equation for \( u_l \) we get

\[
r_{l+1} = \left( -\frac{1}{r_l} + \frac{\Lambda}{s_l} - \frac{X_l}{s_l} \right) \left( 1 + \frac{Y_l}{s_l} \right)^{-1}.
\]

We can write

\[
r_{l+1} = r_l \cdot J_l L_{l,\lambda} W_l,
\]

where

\[
x \cdot J_l = -\frac{1}{x} + \frac{c}{\sqrt{1 - \frac{1}{n}}},
\]

\[
x \cdot L_{l,\lambda} = x + \frac{\lambda}{2\sigma(c)s_l\sqrt{n}},
\]

and

\[
x \cdot W_l = \left( x - \frac{X_l}{s_l} \right) \left( 1 + \frac{Y_l}{s_l} \right)^{-1}.
\]

The notation we have used here is that

\[
r \cdot J_l L_{l,\lambda} W_l = W_l \left( L_{l,\lambda} \left( J_l (r) \right) \right).
\]

The operator \( L_{l,\lambda} \) is infinitesimal (that is, \( x \cdot L_{l,\lambda} - x = O(n^{-1}) \) ) because

\[
x \cdot L_{l,\lambda} = x + \frac{\lambda}{2\sigma(c)s_l\sqrt{n}} \approx x + \frac{\lambda}{2\sigma(c)s(t)n}.
\]

The operator \( W_l \) essentially adds a random variable with variance \( 1/n \) (recall properties of \( X_l \) and \( Y_l \)) and hence it’s also infinitesimal. The operator \( J_l \) is not infinitesimal and the roots of the characteristic equation are exactly the fixed points of \( J_l \). In particular,

\[
q_1 \cdot J_l = q_1.
\]
$J_l$ is an isometry of the hyperbolic plane with a single fixed point. This implies that $J_l$ is a rotation around the fixed point. We will call this point $\rho_l$. Define the map $T_l$ by

$$r.T_l = \frac{r - Re(\rho_l)}{Im(\rho_l)}.$$ 

Hence $\rho_l T_l = i$

Let $J_l = T_l Q_l T_l^{-1}$. If we map the upper half plane to the unit disc $\mathbb{D}$ by the conformal map,

$$U(z) = \frac{i - z}{i + z},$$ \hspace{1cm} (1.5)

then on the disc, the map $Q_l$ is just rotation around 0 by the angle $-2\text{arg}(\rho_l)$. Hence

$$r_{l+1} = r_l T_l Q_l T_l^{-1} L_{l,\lambda} W_l.$$

If we transform the ratios to the space of angles by setting $\phi_l = \text{arg}(U(r_l))$ then we get,

$$\phi_{l+1} = \phi_l T_l Q_l T_l^{-1} L_{l,\lambda} W_l.$$

Let

$$\tilde{\phi}_l = \phi_l T_l G_{l-1}$$

and

$$G_l = Q_l^{-1}Q_l^{-1}\cdots Q_l^{-1}.$$

Then

$$\tilde{\phi}_l = \phi_{l+1} T_{l+1} G_l$$

$$= \phi_l T_l Q_l T_l^{-1} L_{l,\lambda} W_l T_{l+1} G_l$$

$$= \tilde{\phi}_l T_l^{-1} L_{l,\lambda} W_l T_{l+1} G_l$$

Note that

$$(T_l^{-1} L_{l,\lambda} W_l T_{l+1}) = (T_l^{-1} L_{l,\lambda} W_l T_l) (T_l^{-1} T_{l+1}).$$

The operator $L_{l,\lambda} W_l$ is infinitesimal and so the operator $(T_l^{-1} L_{l,\lambda} W_l T_l)$ is also infinitesimal. The operator $(T_l^{-1} T_{l+1})$ is infinitesimal as well and this shows that the evolution of $\tilde{\phi}_l$ takes infinitesimal time steps.

We can use this to derive an SDE for the evolution of $\phi_{l_1} - \phi_{l_2}$ and then after some work it yields the SDE (1.2). Note that we are constrained to have $t < 1 - \frac{\epsilon^2}{4}$ in the discrete Markov chain. Let $n_0 = \left(1 - \frac{\epsilon^2}{4}\right) n$ and $n_1 = n - n_0$. Note that $c\sqrt{n} = 2\sqrt{n_1}$.

Fix an $\epsilon > 0$. Consider the tridiagonal matrix $\tilde{M}_n$ that we obtained in the last lecture. We can break it up into three tridiagonal blocks of sizes $(1 - \epsilon)n$, $\epsilon n$ and $n_1$. We can solve the eigenvalue equation on each triangle block.
One can show that for \( t \approx \frac{n_0(1-\epsilon)}{n} (1-\epsilon) \), \( \alpha_\lambda(t) \) is close to 0 mod 2\( \pi \) with high probability. The second tridiagonal part does not contribute much and \( \alpha_\lambda(t) \) does not change much as \( t \) goes from \( \frac{n_0(1-\epsilon)}{n} \) to \( \frac{n}{\sqrt{n_1}} \). Since \( c\sqrt{n} = 2\sqrt{n_1} \), the third tridiagonal part corresponds to the edge of the eigenvalue spectrum. We know from previous lectures that close to the edge, eigenvalues are spaced roughly distance \( n^{-1/6} \) apart. Therefore when we zoom in by a factor of \( \sqrt{n} \), we will not see anything in the limit with high probability. This completes the derivation of the SDE (1.2) and completes the proof of the scaling limit result. With some more work one can show that the limiting process is independent of \( c \).

2 Applications

2.1 Estimating probabilities

Suppose for the limiting Sine\(_\beta\) distribution, we want to estimate the probability

\[
p_{\lambda,\beta} = P([0, \lambda] \text{ is empty}).
\]

For \( \beta = 1, 2 \) and 4 we have explicit expressions involving Fredholm determinants of the Sine kernel. Now we can estimate \( p_{\lambda,\beta} \) for general \( \beta \). We get the following result.

Lemma 1. \( p_{\lambda,\beta} \sim c_{\beta} \lambda^{\frac{1}{4}} \left( \frac{\beta}{2} + \frac{2}{3} - \frac{3}{2} \right) \exp \left( -\frac{\beta}{4} \lambda^2 + \left( \frac{\beta}{4} - \frac{1}{2} \right) \lambda \right) \),

where \( c_{\beta} \) is a ‘weird’ constant.

Proof. Here we give a sketch of the proof. For a fixed \( \lambda \) the SDE (1.2) reduces to

\[
d\alpha_\lambda = \frac{\beta}{4} e^{-\frac{\beta}{4} t} dt + 2 \sin \left( \frac{\alpha_\lambda}{2} \right) dB,
\]

with the initial condition \( \alpha_\lambda(0) = 0 \).

We know that

\[
\frac{1}{2\pi} \lim_{t \to \infty} \alpha_\lambda(t) \overset{d}{=} \text{Number of eigenvalues in } [0, \lambda].
\]

Hence

\[
P([0, \lambda] \text{ is empty}) = P(\lim_{t \to \infty} \alpha_\lambda(t) = 0)
\]

\[
= P(\alpha_\lambda(t) \in [0, 2\pi) \text{ for all } t)
\]

\[
= P(X(t) \text{ never blows up}),
\]

where

\[
X(t) = \log \left( \tan \left( \frac{\alpha_\lambda(t)}{4} \right) \right)
\]

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and $X(0) = -\infty$. The process $X$ satisfies the following equation

$$dX = \lambda \beta e^{-\frac{\beta}{4} t} \cosh(X) dt + \frac{1}{2} \tanh(X) dt + dB_t.$$ 

Using stochastic calculus we can approximate the blow-up probability with high accuracy.

### 2.2 Random Schrodinger Operators

Random Schrodinger operator looks like

$$\partial_{xx} + \text{“noise”}.$$

We can discretize this operator by expressing $\partial_{xx}$ as a discrete Laplacian

$$\begin{bmatrix}
-2 & 1 \\
1 & -2 & 1 \\
& \ddots & \ddots & \ddots \\
& & & 1 & -2
\end{bmatrix}_{n \times n}$$

and adding to it the noise term which looks like

$$\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}_{n \times n},$$

where $X_i \sim N(0, 1)$.

We can set up the eigenvalue equation as in the proof of the bulk scaling limit for the $\beta$-ensemble and obtain a three term recursion. We can then follow the same steps to get a point process as the scaling limit.

### 3 Recent Developments

For $\beta$ ensembles of eigenvalues we have shown that we get Tracy-Widom distribution as the scaling limit of eigenvalues around the edge and we get the $\text{Sine}_{\beta}$ point process in the bulk. Recent developments have indicated that these results are fairly robust and extend to eigenvalue distributions of general Wigner Hermitian and Wigner Symmetric matrices.

For GUE and GOE, the joint eigenvalue density is of the form of the $\beta$ ensemble

$$p(\lambda) = \Delta(\lambda)^\beta e^{-\frac{\beta}{4} \sum \lambda_i^2}$$
We can write the density function of the matrices itself as
\[
\frac{1}{Z} e^{-\frac{\alpha}{4} \text{Tr}(M^2)} dM.
\]

We can generalize this to
\[
\frac{1}{Z} e^{-\frac{\alpha}{4} V(M)} dM,
\]
where \( V \) is a ‘nice’ function. For a general \( V \) the matrices with such a density function will not have independent entries.

Under this generalization it has been shown for \( \beta = 1, 2, 4 \) by using Riemann-Hilbert techniques that the same scaling gives us the same edge and bulk results (Tracy Widom at the edge and \( Sine_\beta \) in the bulk).

In 1999 Soshnivkov showed by a clever use of the moment method that the same edge result is true for more general matrices with good control over the moments of their entries.

In 2001 Johannson proved the bulk scaling limit result for matrices of the form
\[
M = \epsilon A + B
\]
where \( \epsilon > 0, \) \( A \) is a GUE, \( B \) is a general Wigner matrix and \( A, B \) are independent.

In May 2009, Erdos-Ramirez-Schlein-Yau proved the bulk scaling limit result for Wigner matrices with exponential moments condition on the entries. A week later Tao-Vu proved the same with finite moment condition.