1. A snowboarder slides (without friction) from $(0,0)$ to $(\pi R, -2R)$. This takes time $T_c = \pi \sqrt{R/g}$ if the track is a cycloid (as we calculated in class).

**Key Mathematical concept:** velocity = distance/time. If moving along a curve, distance = arclength, so

$$v = \frac{ds}{dt}$$  \hspace{1cm} (1)

**Key Physical concept:** conservation of energy $v^2/2 + gy = \text{Const.}$, where $y$ points upward, opposite to gravity. Here, we always start at $(0,0)$ with $v = 0$ so the constant is always 0 and

$$\frac{v^2}{2} = -gy$$ \hspace{1cm} (2)

($y$ points upward so we can only move towards $y < 0$. Math makes sense).

Combining (1) and (2)

$$\frac{ds}{dt} = \sqrt{-2gy} \iff dt = \frac{ds}{\sqrt{-2gy}}$$ \hspace{1cm} (3)

It’s now just a matter of cranking it out if you understood arclength and parametrization of curves. Use Maple’s `int` and `evalf` if needed but you must first “scale out” $R$ and $g$ with simple substitutions. All times should be proportional to $\sqrt{R/g}$, it’s the only physical time in the problem.

(a) Straight line: must be $x = -\pi y/2$ so $ds = \sqrt{dx^2 + dy^2} = \sqrt{\pi^2 + 4} |dy|/2$ and the time to go from $(0,0)$ to $(\pi R, -2R)$ along that straight line is (integrating (3)):

$$T_l = \int_{-2R}^{0} \frac{\sqrt{\pi^2 + 4}}{2\sqrt{-2gy}} dy = \sqrt{\pi^2 + 4} \int \frac{R}{g} \approx 3.724 \sqrt{\frac{R}{g}}.$$ \hspace{1cm} (4)

(b) Sinusoid: must be $y = -2R \sin(x/(2R))$ so $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \cos^2(x/(2R))} |dx|$ and

$$T_s = \frac{1}{2\sqrt{gR}} \int_{0}^{\pi R} \frac{\sqrt{1 + \cos^2 \frac{x}{2R}}}{\sqrt{\sin \frac{x}{2R}}} dx = \frac{R}{g} \int_{0}^{\pi/2} \frac{1 + \cos^2 u}{\sqrt{\sin u}} du \approx 3.3636 \sqrt{\frac{R}{g}}.$$ \hspace{1cm} (5)

where the last integral is done with Maple’s `evalf`.

(b) Parabola: must be $x = \pi y^2/(4R)$ so $ds = \sqrt{dx^2 + dy^2} = \sqrt{\pi^2 y^2 + 4R^2} |dy|/(2R)$ and

$$T_p = \frac{1}{2R\sqrt{2g}} \int_{0}^{2R} \frac{\sqrt{4R^2 + \pi^2 y^2}}{\sqrt{y}} dy = \frac{R}{g} \int_{0}^{1} \frac{1 + \pi^2 u^2}{\sqrt{u}} du \approx 3.1724 \sqrt{\frac{R}{g}}.$$ \hspace{1cm} (6)

Summary: $T_l > T_s > T_p > T_c$, cycloid is fastest, line is slowest. Try it at Tyrol basin!

2. See picture p. 661. You may follow the hints there and guess that $r = Ce^{-\theta}$ but need to thoroughly show that this is correct. (Briefly, by symmetry the angle between the radial line joining the center to an ant and the ant direction is always $\pi/4$, so trajectory must be an
equiangular spiral (we saw that one before), should show that \( r = Ce^{-\theta} \) indeed as this \( \pi/4 \) equi-angular property.

More general approach: take Cartesian axes centered at the center of the square. The center does not move. Coordinates of one ant are \((x_1, y_1) \equiv (x, y)\) so the coordinates \((x_2, y_2)\) of the next ant in the counterclockwise direction are rotated by \(\pi/2\), that’s \((x_2, y_2) = (-y, x)\). Now the ant at \((x_1, y_1)\) moves in the direction of the ant at \((x_2, y_2)\) i.e.

\[
\frac{dy}{dx} = -\frac{y_1 - y_2}{x_1 - x_2} = \frac{y - x}{y + x}.
\]

This is a scale-invariant differential equation and could be solved as seen in 15.1. However it is easier in this case to switch to polar coordinates with \(x = r \cos \theta, y = r \sin \theta\) and

\[
\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta.
\]

Then by the chain rule \(dy/dx = (dy/d\theta)/(dx/d\theta)\), so after some simple algebra

\[
\frac{dr}{d\theta} = -r \quad \Rightarrow \quad r = Ce^{-\theta}.
\]

Finally at \(\theta = \pi/4, r = L/\sqrt{2}\) so \(C = Le^{\pi/4}/\sqrt{2}\). Compute the length by simple application of arclength in polar coordinates integrating from \(\theta = \pi/4\) to \(\theta = \infty\).

3. Stewart 10.2 # 70. Obviously some Geometric Series business. If \(R_i\) is the radius of a circle and \(D_i\) is the distance of its center from a vertex, then \(R_i = D_i \sin \pi/6\), true for any circle. For the biggest circle: \(R_1 = 1/(2\sqrt{3})\). The size of the next circle toward the vertex is determined from the equality \(D_{i+1} + R_{i+1} = D_i - R_i\) (do you see where that comes from?). So \(R_{i+1} = R_i/3\) and \(A = 11\pi/96 \approx 0.35997,\) The area of the triangle is \(\sqrt{3}/4 \approx 0.43301\).

4. Calculate \(\sin 31^\circ\) (sine of 31 degrees) to 6 decimal places using only the 4 basic operations +, −, *, /. Obviously some Taylor Series application.

**TRAP 1:** work with those useless degrees! In radians \(31^\circ \equiv 31\pi/180\) \(rad = \pi/6 + \pi/180\). Then by Taylor’s Formula:

\[
\sin x = \sin a + \cos a (x - a) - \sin a \frac{(x - a)^2}{2} - \cos a \frac{(x - a)^3}{3!} + \sin a \frac{(x - a)^4}{4!} + \cdots + R_n
\]

where we know some good things about the remainder \(R_n\) (see page 636) namely in this sine case \(|R_n| < |x - a|^{n+1}/(n + 1)!|\).

**TRAP 2:** Expand about \(a = 0\). Smarter to pick \(a = \pi/6\), for which basic geometry gives \(\sin a = 1/2\) and \(\cos a = \sqrt{3}/2\). We should safely get 6 digits if \(R_n < 10^{-7}\) i.e. \((\pi/180)^{n+1} < (n + 1)!/10^7\), this is true for \(n \geq 3\). The \(n = 3\) approximation is

\[
\sin \left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2} \frac{\pi}{180} - \frac{1}{4} \left(\frac{\pi}{180}\right)^2 - \frac{\sqrt{3}}{12} \left(\frac{\pi}{180}\right)^3 \approx 0.51503807296522
\]

which is actually correct to 8 digits. (BTW, what is the limit of the sequence \(a_1 = 1.5, \ a_{n+1} = 0.5(a_n + 3/a_n)\) calculate a few terms. See #23 in Sect. 2.10. Could you have competed with the ancient Babylonians?)