Detailed discussion of ‘charged particle moving in a constant magnetic field,’ or in mathematical terms
\[
\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{b} \times \vec{v}
\]
(1)

where \(\vec{v} = \vec{v}(t) = d\vec{r}/dt\).

**Physics remark:** we have absorbed all the scalar constants like electric charge \(q\) and mass \(m\) into the \(\vec{b}\). You can think of \(\vec{b}\) as the ‘magnetic field’ but it’s actually the magnetic field times some constants that have their own physical units. If you know about the Lorentz force from E&M then \(\vec{b} = (−q/m)\vec{B}\). What are the physical units of \(\vec{b}\) in equation (1)? [Hint: the equation tells you!]

**Goal:** Equation (1) is the vector differential equation for \(\vec{v}(t)\). We want to solve it for \(\vec{v}(t)\). The constant vector \(\vec{b}\) is ‘known’.

The first thing we notice is that \(d\vec{v}/dt = \vec{b} \times \vec{v}\) is always perpendicular to both \(\vec{b}\) and \(\vec{v}\).

\(\triangleright\) Perpendicularity of \(\vec{v}\) and \(d\vec{v}/dt\) means that
\[
2\vec{v} \cdot \frac{d\vec{v}}{dt} = 0 = \frac{d|\vec{v}|^2}{dt}
\]
(2)
(because \(|\vec{v}|^2 = \vec{v} \cdot \vec{v}\) and \(d(\vec{v} \cdot \vec{v})/dt = \dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}} = 2\dot{\vec{v}} \cdot \vec{v}\)). So the magnitude of \(\vec{v}\) does not change but its direction changes.

\(\triangleright\) Next, ‘dot’ the equation (1) with \(\hat{\vec{b}}\) since \(\vec{b} \times \vec{v}\) is also perpendicular to \(\vec{b}\), and
\[
\hat{\vec{b}} \cdot \frac{d\vec{v}}{dt} = \hat{\vec{b}} \cdot (\vec{b} \times \vec{v}) = 0 \Rightarrow \frac{d}{dt} (\hat{\vec{b}} \cdot \vec{v}) = 0
\]
(3)

(we can take the \(\hat{\vec{b}}\) inside the \(d/dt\) because the \(\vec{b}\) direction is constant since \(\vec{b}\) is constant). This is telling us that the component of \(\vec{v}\) in the direction of \(\vec{b}\) does not change.

This is a big step because it says that all the ‘action’ is happening in the plane perpendicular to \(\vec{b}\).

So write \(\vec{v} = \vec{v}_\parallel + \vec{v}_\perp\), where \(\vec{v}_\parallel = \hat{\vec{b}} (\hat{\vec{b}} \cdot \vec{v})\) is constant as shown in (3), then eqn (1) implies
\[
\frac{d\vec{v}_\perp}{dt} = \vec{b} \times \vec{v}_\perp.
\]
(4)

You can picture this more easily. Take \(\vec{b}\) perpendicular to your sheet of paper, then \(\vec{v}_\perp\) and \(\vec{b} \times \vec{v}_\perp = d\vec{v}_\perp/dt\) are all in the plane of the paper (think about it! draw it!). As before this equation means that \(|\vec{v}_\perp|\) is a constant (why was that again?). So we know almost everything about \(\vec{v}(t)\), its component parallel to \(\vec{b}\) does not change and its component perpendicular to \(\vec{b}\) has constant magnitude. Constant magnitude makes us think ‘circle’. What ‘circle’ though?

(1) **Smart component approach:** To solve (4) take any fixed unit vectors \(\hat{\vec{x}}\) and \(\hat{\vec{y}}\) such that \(\hat{\vec{x}}, \hat{\vec{y}}\) and \(\vec{b}\) form a right-handed orthonormal basis, then let \(\vec{v}_\perp = x(t)\hat{\vec{x}} + y(t)\hat{\vec{y}}\) (why can we write that?). Since \(\hat{\vec{x}}\) and \(\hat{\vec{y}}\) are fixed, (4) becomes
\[
\frac{d\vec{v}_\perp}{dt} = \dot{x} \hat{\vec{x}} + \dot{y} \hat{\vec{y}} = \vec{b} \times \vec{v}_\perp = |\vec{b}| (x \dot{\hat{\vec{y}}} − y \dot{\hat{\vec{x}}})
\]
(5)
(why?) which splits into the two coupled ODEs (why?)

\[ \dot{x} = -\omega y, \quad \dot{y} = \omega x \]

(6)

where \( \omega = |\vec{b}| \) is a (positive) constant. You should make a mental note of that system, it’s a fundamental one. You can eliminate one of the unknown functions, \( y(t) \) say, to get \( \dot{x} + \omega^2 x = 0 \). This is the harmonic oscillator equation. Its two linearly independent solutions are \( x(t) = \cos \omega t \) and \( x(t) = \sin \omega t \) and the general solution is \( x(t) = A \cos \omega t + B \sin \omega t \) for some constants \( A \) and \( B \) to determine from the initial conditions. We can also write the general solution in the form \( x(t) = \omega \cdot \vec{b} \), and the direction of the solution in the form \( x(t) = \omega \cdot \vec{b} \), where \( \vec{b} \) and the angle between \( \vec{b} \) and \( \vec{b} \) is constant. You should make a mental note of that system, it’s a harmonic oscillator equation. Its two linearly independent solutions are \( \cos \omega t \) and \( \sin \omega t \) for some constants \( A \) and \( B \) to determine from the initial conditions. We can also write the general solution in the form \( x(t) = C \cos(\omega t - t_0) \) and take \( C \) and \( t_0 \) as the ‘constants of integration’. But if we write it in that form then \( x_0 = x(t_0) = C \) and \( \dot{x}(t_0) = 0 \) which implies \( y(t_0) = 0 \) from (6). So we can write the solution in the form \( x(t) = x_0 \cos(\omega t - t_0) \) and from the first equation in (6), \( y(t) = x_0 \sin(\omega t - t_0) \), for some constants \( x_0 \) and \( t_0 \). But this choice of constants of integration means that \( \vec{v}_\perp(t_0) = x_0 \hat{x} \) since by definition \( \vec{v}_\perp(t) = x(t)\hat{x} + y(t)\hat{y} \). So why bother with those \( \hat{x} \) and \( \hat{y} \) at all? we can write

\[ \vec{v}_\perp(t) = \cos \varphi \vec{v}_\perp(t_0) + \sin \varphi \left( \vec{b} \times \vec{v}_\perp(t_0) \right) \]

(7)

where \( \varphi = |\vec{b}|(t - t_0) \). (Substitute back into (4) to check and digest the solution (7)).

(2) Direct vector approach: We can actually deduce (7) ‘directly’ from eqn (4) since the latter implies that \( |\vec{v}_\perp(t)| = |\vec{v}_\perp(t_0)| \) (i.e. the magnitude is constant). Furthermore the magnitude of \( \vec{b} \) is constant and the angle between \( \vec{b} \) and \( \vec{v}_\perp \) is constant (it’s \( \pi/2 \)). So \( |\vec{b} \times \vec{v}_\perp| = |\vec{b}| \cdot |\vec{v}_\perp| = \text{constant} \) and the direction of \( \vec{b} \times \vec{v}_\perp \) is always \( \pi/2 \) counterclockwise around \( \vec{b} \) from that of \( \vec{v}_\perp \) as shown in the following figure on the left.

But this clearly means that \( \vec{v}_\perp \) is rotating counterclockwise about \( \vec{b} \) at angular velocity \( |\vec{b}| \). So if we put the tail of the velocity vector \( \vec{v}_\perp(t) \) at the same point for all \( t \), then its head moves along a circle of radius \( |\vec{v}_\perp(t_0)| \) at angular velocity \( |\vec{b}| \). We can use \( \vec{v}_\perp(t_0) \) as one basis vector and \( \vec{b} \times \vec{v}_\perp(t_0) \) as the other orthogonal vector, and write the solution (7) directly with \( \varphi = |\vec{b}|(t - t_0) \). This should be obvious from the figure if you remember your basic trigonometry.
Putting it all together: remember that \( \vec{v}_\parallel(t) = \vec{v}_\parallel(t_0) \) so we could actually write the full solution to (1) in vector form as
\[
\vec{v}(t) = \vec{v}_\parallel(t_0) + \cos \varphi \, \vec{v}_\perp(t_0) + \sin \varphi \left( \vec{b} \times \vec{v}_\perp(t_0) \right)
\]  
but we can also clean this up to express it in terms of the full \( \vec{v}(t_0) = \vec{v}_0 \). How? well, first \( \vec{v}_\parallel = \vec{b}(\vec{b} \cdot \vec{v}) \), then \( \vec{v}_\perp = \vec{v} - \vec{v}_\parallel \) and \( \vec{b} \times \vec{v}_\perp = \vec{b} \times \vec{v} \) (why?), so (8) can be written
\[
\vec{v}(t) = \vec{b}(\vec{b} \cdot \vec{v}_0) + \cos \varphi \left( \vec{v}_0 - \vec{b}(\vec{b} \cdot \vec{v}_0) \right) + \sin \varphi \left( \vec{b} \times \vec{v}_0 \right)
\]  
A bit scary at first, but this is a very clean and general way to write the solution. Everything is a constant in there except \( \varphi = |\vec{b}|(t - t_0) \). Note that each term has units of velocity as should be (what are the units of \( \varphi \)?)

Now if we want to get the particle trajectory \( \vec{r}(t) \) we’re in pretty good shape since we know \( \vec{v}(t) = d\vec{r}/dt \) explicitly. We only need to integrate (9) with respect to \( t \) to obtain
\[
\vec{r}(t) = t \vec{b}(\vec{b} \cdot \vec{v}_0) + \frac{1}{|\vec{b}|} \sin \varphi \left( \vec{v}_0 - \vec{b}(\vec{b} \cdot \vec{v}_0) \right) - \frac{1}{|\vec{b}|} \cos \varphi \left( \vec{b} \times \vec{v}_0 \right) + \vec{r}_c \]  
where \( \vec{r}_c \) is a constant of integration which we could figure out in terms of the position at \( t_0 \) (is \( \vec{r}_c = \vec{r}(t_0) \)?) This is the equation of a helix whose axis is parallel to \( \vec{b} \). The first term is the uniform (constant speed) motion along the direction \( \vec{b} \), the next two terms are the rotation at rate \( |\vec{b}| \) in the directions perpendicular to \( \vec{b} \), and the constant term \( \vec{r}_c \) is a point on the axis of rotation for that particle as a function of its initial conditions. Exercises: Find \( \vec{r}_c \) in terms of the position \( \vec{r}_0 \) and velocity \( \vec{v}_0 \) at time \( t_0 \). What is the equation of the axis of the helix?

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IMPORTANT REMARKS, EXTENSIONS AND ANALOGIES

All of this is quite fundamental because the underlying math is not just for the motion of a charged particle in a constant magnetic field. We get the same equation if we have uniform rotation about an axis. For instance if a particle rotates at constant angular velocity \( \omega \) counterclockwise about an axis parallel to the direction \( \vec{n} \) (unit vector) that passes through the origin, then its velocity is
\[
\vec{v} = \vec{\omega} \times \vec{r} \iff \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}
\]
where \( \vec{\omega} = \omega \vec{n} \). This is the same equation as (1) but with \( \vec{r} \) instead of \( \vec{v} \) and \( \vec{\omega} \) instead of \( \vec{b} \). The solution of this differential equation for \( \vec{r}(t) \) is that the component of \( \vec{r} \) in the \( \vec{\omega} \) direction does not change, and the perpendicular component rotates counterclockwise about the \( \vec{n} \) axis at rate \( \omega \). So mathematically it is the same problem although in the Lorentz force case it’s the velocity that is ‘rotating about an axis’ (and as a result the particle moves along a helix while here it is the particle which rotates about an axis passing through the origin and the particle moves along a circle.)

More generally if \( \vec{u}(t) \) is any vector function of \( t \) and if its equation is
\[
\frac{d\vec{u}}{dt} = \vec{c} \times \vec{u}
\]
where \( \vec{c} \) is constant, then the solution \( \vec{u}(t) \) corresponds to the rotation of \( \vec{u} \) counterclockwise about the direction \( \vec{c} \) at angular rate \( |\vec{c}| \).
Generalizing a bit, if a particle rotates at $\omega$ about an axis parallel to $\vec{n}$ that goes not through the origin but through a point $\vec{r}_a$ then its velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times (\vec{r} - \vec{r}_a) \quad (13)$$

Indeed you easily check that $|\vec{r} - \vec{r}_a| = \text{constant}$ (how do you ‘easily check’ that?) and that you can rewrite the equation in the form

$$\frac{d}{dt}(\vec{r} - \vec{r}_a) = \vec{\omega} \times (\vec{r} - \vec{r}_a) \quad (14)$$

since $\vec{r}_a$ is a constant (but $\vec{r} = \vec{r}(t)$ is not constant, and $|\vec{r}|$ is not constant either).

This brings us back to (1) which in Newton’s dot notation reads $\ddot{\vec{r}} = \vec{b} \times \dot{\vec{r}}$ but since $\vec{b} = \text{constant}$ then $d(\vec{b} \times \vec{r})/dt = \vec{b} \times \dot{\vec{r}}$ and we can integrate the equation once in time to obtain

$$\frac{d\vec{r}}{dt} = \vec{b} \times \vec{r} + \vec{c} \quad (15)$$

where $\vec{c}$ is a constant of integration which is in fact a velocity. This looks a lot like (13) except for that (vector) constant $\vec{c}$. But we could write it in the form (13) by simply defining $\vec{c} = -\vec{b} \times \vec{r}_a$. Can we really do that? $\vec{c}$ is an arbitrary constant of integration that would be determined by initial conditions, since $\vec{b}$ is known we can in principle find $\vec{r}_a$ in terms of whatever $\vec{c}$ is. Well, not quite. Remember the problem $\vec{a} \times \vec{x} = \vec{b}$ (section 1.6.3) that Megan Sharrow solved for us. The vector $\vec{b} \times \vec{r}_a$ has to be perpendicular to $\vec{b}$, but $\vec{c}$ does not have to. So we cannot always find a $\vec{r}_a$ such that $\vec{b} \times \vec{r}_a = -\vec{c}$ for any $\vec{c}$. But we can find a $\vec{r}_a$ and a $\vec{c}_\parallel$ parallel to $\vec{b}$ such that

$$\vec{c} = \vec{c}_\parallel - (\vec{b} \times \vec{r}_a). \quad (16)$$

so we can always write (15), the first integral of (1), in the form

$$\frac{d\vec{r}}{dt} = \vec{b} \times (\vec{r} - \vec{r}_a) + \vec{c}_\parallel \quad (17)$$

Now the educated reader says ‘Oh! the velocity is simply the sum of a rigid body rotation about an axis parallel to $\vec{b}$ passing through $\vec{r}_a$ and a uniform translation at speed $\vec{c}_\parallel$ in the direction of $\vec{b}$. ’ You want to think, digest and practice all this to get to the point where you readily ‘see’ what equation (17) means.

**Exercise:** Find $\vec{r}_a$, $\vec{c}_\parallel$ and the radius of the cylinder on which the particle moves (the gyroradius) in terms of the particle position $\vec{r}_0$ and velocity $\vec{v}_0$ at time $t_0$. Work from eqn (17) *not* from (10).

So the equations (1), (12) and (14) are key ‘building blocks’. The solution (9) is also quite important for other applications.

Suppose you were given this problem: rotate vector $(4, 1, 6)$ by $\pi/3$ about the direction $(1, 2, 3)$. How would you tackle that?!!

(This could be part of a computer graphic problem: you have a bunch of edges representing a structure for instance and you want to rotate all these edges about an axis, for visualization purposes).
Well, if $\vec{v}_0$ is the vector you want to rotate and $\vec{b}$ is the rotation direction then equation (9) gives you the general formula that does the trick for any angle $\varphi$!

$\triangleright$ So what’s the rotation of $(4, 1, 6)$ by $\pi/3$ about $(1, 2, 3)$?! figure it out!