1. [20pts] Specify whether the following problems are well-posed and give the name of the equation if you recognize it. If the equation admits wave solutions, state whether or not the waves are dispersive. In all cases $-\infty < x < \infty$, $0 < t < \infty$.

Fourier analysis: $u(x, t) = e^{\sigma t} e^{ikx} \equiv e^{-i\omega t} e^{ikx}$

(a) $u_t = u_{xx}$ with $u(x, 0) = f(x)$.
   $$\sigma = -k^2$$ well-posed. Heat equation. Diffusion, no waves.

(b) $u_{tt} = u_{xx}$ with $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$.
   $$\sigma \equiv -i\omega = \pm ik$$, well-posed. Wave equation, not dispersive: $\omega/k = \pm 1$, $d\omega/dk = \pm 1$, all waves travel at same speed. (Two waves, one travels left, the other right, both at speed 1).

(c) $u_{ttt} = u_{xx}$ with $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $u_t(x, 0) = h(x)$.
   $$\sigma^3 = -k^2$$, so $\sigma = |k|^{2/3} \{-1, e^{i\pi/3}, e^{-i\pi/3}\}$, ill-posed as two of the $\sigma$'s have positive real parts $= |k|^{2/3}/2$, which is unbounded as $k \to \infty$.

(d) $u_t = iu_{xx}$ with $u(x, 0) = f(x)$, where $i^2 = -1$.
   $$\sigma = -ik^2 \equiv -i\omega$$, so $\omega = k^2$. Well-posed, dispersive waves. Phase velocity $= \omega/k = k$ so short waves travel faster. This is Schrödinger’s equation (quantum mechanics, particles $\equiv$ waves).

(e) $u_{tt} = u_{xx} + u$ with $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$.
   $$\sigma^2 = 1 - k^2$$, $\sigma = \pm \sqrt{1 - k^2}$. Well-posed but unstable for $|k| < 1$. Exponential growth if $|k| < 1$ (instability), dispersive waves if $|k| > 1$.

2. [20pts] Consider the PDE $u_t + u_{xx} + u_{xxxx} = 0$ with $u(x, 0) = f(x)$. 

(a) Give an integral representation for the solution $u(x, t)$ in $-\infty < x < \infty$, $t > 0$ and specify the asymptotic form of the solution for large $t$. Assume $\int_{-\infty}^{\infty} |f(x)|dx \leq M < \infty$ in this case.

$$u(x, t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{(k^2 - k^4)t} e^{ikx} dk$$, where $\hat{f}(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$ (Fourier transform).

Now, $\int_{-\infty}^{\infty} |f(x)|dx \leq M < \infty$ implies $|\hat{f}(k)| \leq M/(2\pi)$. Thanks to the $e^{-k^4t}$ the $u(x, t)$ integrand is well-behaved as $|k| \to \infty$ (for $t > 0$). (And the equation is well-posed).

For large $t$, the integral is increasingly dominated by the neighborhoods of $\max(k^2 - k^4)$, i.e. $k = \pm 1/\sqrt{2}$. So for large $t$, with $a = 1/\sqrt{2}$ and $\epsilon \ll 1$,

$$u(x, t) \approx \int_{-a-\epsilon}^{-a+\epsilon} \hat{f}(k) e^{(k^2 - k^4)t} e^{ikx} dk + \int_{a-\epsilon}^{a+\epsilon} \hat{f}(k) e^{(k^2 - k^4)t} e^{ikx} dk$$

$$\approx (\hat{f}(-a) e^{-iax} + \hat{f}(a) e^{iax}) e^{t/4} \int_{-\infty}^{\infty} e^{-2s^2t} e^{isx} ds$$

$$= (\hat{f}(-a) e^{-iax} + \hat{f}(a) e^{iax}) \sqrt{\frac{\pi}{2t}} e^{t/4} e^{-x^2/8t}.$$
(b) What is the solution and its asymptotic form for large \( t \) if \( f(x) = f(x + 2\pi) \)?

Function is periodic, so its Fourier transform does not exist (it’s a sum of “delta functions”), but its Fourier series does

\[
    u(x, t) = \sum_{n=-\infty}^{\infty} C_n e^{inx} e^{(n^2-n^3)t}.
\]

For large \( t \), \( u(x, t) \approx C_0 + C_1 e^{ix} + C_{-1} e^{-ix} \), steady! no exponential growth as in the unbounded case.

3. [10pts] Solve \( u_{tt} = u_{xx} + \delta(x - Vt) \) with \( u(x, t), u_t(x, t) \to 0 \) as \( t \to -\infty \) and \( V > 1 \).

Comment on the case \( V = 1 \).

The Green’s function of the 1D wave equation is (cf. class notes)

\[
    G(x, t; x_0, t_0) = \frac{1}{2} H(t - t_0) [H(x - x_0 + t - t_0) - H(x - x_0 - t + t_0)].
\]

So the solution to \( u_{tt} = u_{xx} + F(x, t) \) with \( u(x, t), u_t(x, t) \to 0 \) as \( t \to -\infty \) is

\[
    u(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x_0, t_0) G(x, t; x_0, t_0) dx_0 dt_0.
\]

Here \( F(x_0, t_0) = \delta(x_0 - Vt_0) \), so

\[
    u(x, t) = \frac{1}{2} \int_{-\infty}^{t} [H(x - Vt_0 + t - t_0) - H(x - Vt_0 - t + t_0)] dt_0.
\]

Note that \( x - Vt_0 + t - t_0 \geq x - Vt_0 - t + t_0, \forall t \geq t_0 \). Now \( H(a) = 0 \) if \( a < 0 \), \( H(a) = 1 \) if \( a > 0 \). So the integral vanishes unless

\[
    x - Vt_0 + t - t_0 > 0 > x - Vt_0 - t + t_0,
\]

or, using \( V > 1 \),

\[
    \frac{x + t}{V + 1} > t_0 > \frac{x - t}{V - 1}.
\]

This requires

\[
    \frac{x + t}{V + 1} > \frac{x - t}{V - 1} \iff Vt > x.
\]

In that case, \( t > (x + t)/(V + 1) \) and

\[
    u(x, t) = \frac{1}{2} \int_{(x-t)/(V-1)}^{(x+t)/(V+1)} dt_0 = \frac{1}{2} \left( \frac{x + t}{V + 1} - \frac{x - t}{V - 1} \right) = \frac{Vt - x}{V^2 - 1},
\]

if \( Vt > x \) and \( u(x, t) = 0 \) if \( Vt < x \).

This problem could be solved as a homogeneous IVP with appropriate initial conditions given on the line \( x = Vt \). If \( V = 1 \), this is equivalent to imposing initial conditions along a characteristic curve, and that’s not good.