This is a Take Home Exam. You must work on it alone. You may use your class notes and the reference books.

1. Solve \( u_t = u_{xx} \) in \( 0 < x < L \) with \( u(0,t) = \partial_x u(L,t) = 0 \) and \( u(x,0) = f(x) \) using (a) a series solution, (b) the Green’s function obtained by the method of images. Compare the two solutions (which will contain integrals involving the initial data \( f(x) \)).

2. Solve \( u_t = u_{xx} \) in \( 0 < x < L \) with \( u(x,0) = f(x) \) and \( u(0,t) = g(t), u(L,t) = 0 \). Provide a general formula in terms of \( f(x) \) and \( g(t) \) then specify the explicit solution when \( f(x) = 1 \) and \( g(t) = e^{-t} \).

3. Find the Green’s function for the biharmonic operator \( \nabla^2 \nabla^2 \) in the plane then solve \( \nabla^2 \nabla^2 u = f(x,y) \) with \( u \) and its derivatives vanishing as \( x^2 + y^2 \to \infty \) and \( f(x,y) \) is smooth with compact support (i.e. vanishes outside of a bounded region).

4. Solve \( e^x u_x + u_y = 0 \) with \( u(x,0) = x \).

5. Solve the traffic flow problem \( \rho_t + q_x = 0 \) for initial conditions corresponding to a traffic light at \( x = 0 \) that turns from red to green at \( t = 0 \). The total number of cars that was waiting at the light is \( N < \infty \). The car flux \( q = \rho V \) where \( V(\rho) \) is a quadratic function with \( V(0) = V_{\text{max}}, V'(0) = 0 \) and \( V(\rho_{\text{max}}) = 0 \).

6. Solve \( (u_x)^2 + u_y + u = 0 \) with \( u(x,0) = x \).