Problem 1 (10 points)

(a) Consider the matrices \[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
0 & 2
\end{bmatrix}, \text{ and }\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix},
\]
as vectors in the vector space \(M_{22}\) of all \(2 \times 2\) matrices.
Do these matrices span \(M_{22}\)?

ANSWER:
These three vectors cannot span \(M_{22}\): We have seen (e.g. in one of the homework problems) that this is a space of dimension 4, so no set of fewer than 4 vectors could span it.

(b) Now use the matrices \[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
0 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}, \text{ and }\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}.
\]
Are these four vectors in \(M_{22}\) linearly independent?

ANSWER:
You could write a lot solving equations, but it is also apparent if you happen to look at them “right” that the third matrix is the sum of the first and last. So one of the matrices is a linear combination of some of the others, so they cannot be linearly independent.

Problem 2 (12 points)
Let \(S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}\) be a set of \(k\) vectors in some vector space \(V\). Prove that \(S\) is linearly dependent if and only if (at least) one of the vectors \(\vec{v}_j\) can be written as a linear combination of all the other vectors in \(S\).

ANSWER:
Here we have to prove the fact that I used in answering problem 1(b)!
Assume the vectors in \(S\) are linearly dependent. That means there is some combination \(a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_k \vec{v}_k = \vec{0}\) giving the zero vector, without all of the coefficients \(a_j\) being zero. Hence in particular \(a_j \neq 0\) for some \(j\). Rearranging the terms, 

\[-a_j \vec{v}_j = (\text{the sum of all the terms } a_i \vec{v}_i \text{ except for } i = j).\]

But since \(a_j \neq 0\) (and hence \(-a_j \neq 0\)) we can multiply by \(1/(-a_j)\) and get 

\[\vec{v}_j = -\frac{a_1}{a_j} \vec{v}_1 - \frac{a_2}{a_j} \vec{v}_2 + \cdots -\frac{a_k}{a_j} \vec{v}_k,\]

where the sum on the right includes terms for each \(v_i\) except \(v_j\). So \(\vec{v}_j\) can be written as a linear combination of the other vectors.

Now assume \(\vec{v}_j\) can be written as a linear combination of the others, and we have to show the vectors are linearly dependent. We have 

\[\vec{v}_j = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_k \vec{v}_k\]

where the right-hand sum extends over all subscripts from 1 to \(k\) except \(j\) and the coefficients \(a_i\) are some numbers. We can add \(-\vec{v}_j\) to both sides and arrange the terms to get 

\[\vec{0} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots - \vec{v}_j + \cdots + a_k \vec{v}_k,\]

where the coefficients are the same as above except that we now have a term \((\text{-1})\vec{v}_j\). This is a linear combination of the vectors in \(S\) that gives \(\vec{0}\): Many of those coefficients might be zero, but the coefficient \(-1\) on \(\vec{v}_j\) is definitely not zero, so this is a linear combination giving \(\vec{0}\) with not all of the coefficients equal to zero, so the vectors are linearly dependent.

Problem 3 (10 points)
Let 

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
2 & 0 \\
2 & 0
\end{bmatrix}, \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \text{ and }\begin{bmatrix}
5 & 5 \\
2 & 2
\end{bmatrix}.
\]
Find a basis for \( \mathbb{R}^3 \) consisting of some of the vectors \( \vec{v}_1 \ldots \vec{v}_5 \).

**ANSWER:**

I will write the vectors as the columns of a matrix, \( A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \). Now row reduce \( A \) and get either (in Row Echelon Form, where there might be variations in this matrix) \( A_R = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \) or (in Reduced Row Echelon Form) \( A_{RR} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \). In either case we see that the first, third, and fourth columns contain leading entries, so we use the first, third, and fourth vectors \( v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \), \( v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \), and \( v_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) from the original set for our basis.

(This is a procedure you could use mechanically. You could also have seen that \( v_2 \) is twice \( v_1 \), so throw out \( v_2 \), and proceeded like that discarding vectors that are linear combinations of the ones you don’t throw out.)

**Problem 4**  
(12 points)

Let \( V \) be any vector space. Suppose for some vector \( \vec{u} \in V \) and some number \( c \), \( c \odot \vec{u} = \vec{0} \), where \( \odot \) is whatever the scalar multiplication is on \( V \) and \( \vec{0} \) is whatever is the zero vector for the space \( V \).

Prove that either \( c = 0 \) (the real number 0) or \( \vec{u} = \vec{0} \).

(This is part (c) of Theorem 4.2: Parts (a) and (b) were that \( 0 \odot \vec{u} = \vec{0} \) and \( c \odot \vec{0} = \vec{0} \) for any vector \( \vec{u} \) and for any number \( c \). You may use those facts in your proof for this problem if you find them helpful.)

**ANSWER:**

Suppose \( c \odot \vec{u} = \vec{0} \) and \( c \neq 0 \): We need to show that in this case \( \vec{u} = \vec{0} \). Since \( c \neq 0 \), there is a number \( \frac{1}{c} \) such that \( \frac{1}{c} \cdot 1 = 1 \). Multiply that number on both sides of our equation \( c \odot \vec{u} = \vec{0} \), getting \( \frac{1}{c} \cdot (c \odot \vec{u}) = \frac{1}{c} \odot \vec{0} \). By an earlier part of this same theorem, \( \frac{1}{c} \odot \vec{0} = \vec{0} \). By the defining requirements for a vector space, \( \frac{1}{c} \odot \vec{u} = \left( \frac{1}{c} \right) \odot \vec{u} \), or \( \frac{1}{c} \odot (c \odot \vec{u}) = 1 \odot \vec{u} \), and by another of the defining requirements for a vector space \( 1 \odot \vec{v} = \vec{v} \) for any vector \( \vec{v} \). Putting these pieces together, \( \vec{u} = 1 \odot \vec{u} = \left( \frac{1}{c} \right) \odot (c \odot \vec{u}) = \frac{1}{c} \odot \vec{0} = \vec{0} \) and we are through.

**Problem 5**  
(10 points)

(a) Show that the set of vectors \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) in \( M_{22} \) which satisfy \( a + b + c + d = 0 \) is a subspace of \( M_{22} \).

**ANSWER:**

It is clear from the way it is described that the set in question is a subset of \( M_{22} \), so we just have to show (i) the set is not empty, (ii) it is closed under addition, and (iii) it is closed under scalar multiplication.

Since \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is in the set, the set is not empty. For (ii), suppose \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( \begin{bmatrix} e & f \\ g & h \end{bmatrix} \) are both in the set, i.e. \( a + b + c + d = 0 \) and \( e + f + g + h = 0 \). Their sum is \( \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix} \), so to determine whether that sum is in our set we ask whether \( (a + e) + (b + f) + (c + g) + (d + h) \) is 0. Rearranging that as \( (a + b + c + d) + (e + f + g + h) = 0 + 0 = 0 \) we see the sum is in
the set. For (iii) we ask a similar question: If \( r \) is any real number and \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\] is in our set, i.e. \( a + b + c + d = 0 \), is \[
\begin{bmatrix}
ra & rb \\
rc & rd
\end{bmatrix}
\] in the set, i.e. is \( ra + rb + rc + rd = 0 \)? But \( ra + rb + rc + rd = r(a + b + c + d) = r \times 0 = 0 \), so the set is closed under scalar multiplication and we are through.

(b) Show that the set of vectors \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\] in \( M_{22} \) which satisfy \( a + b + c + d = 1 \) is not a subspace of \( M_{22} \).

**ANSWER:**

We could test this set against (i) and (ii) and (iii) above. It is definitely not empty. But if we use a more sophisticated version of that test, not just checking to see if some vector is in the set but specifically whether the zero vector \( \vec{0} \) of the space is in the set, we see that it fails: The zero vector in \( M_{22} \) is \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\] and the sum \( 0 + 0 + 0 + 0 = 0 \neq 1 \), so the zero vector is not in the set and hence it cannot be a subspace.

You could also check and find that it fails both (ii) and (iii), but finding that it does not contain \( \vec{0} \) is probably easier.

**Problem 6**  (10 points)

Let \( A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \).

(a) Find a basis for the null space of \( A \), the space of solutions of \( A \vec{x} = \vec{0} \).

**ANSWER:**

In the answer to problem 3 I found that the reduced row echelon form of this matrix is \[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]. Hence the solution space is the set of solutions of \( x_1 + 2x_2 + 3x_5 = 0, \\
x_3 - 2x_5 = 0, \) and \( x_4 + 4x_5 = 0 \). Those can be rewritten as \( x_1 = -2x_2 - 3x_5, x_3 = 2x_5, \) and \( x_4 = -4x_5 \). We can choose any arbitrary values for \( x_2 \) and \( x_5 \) and use those equations to determine values for \( x_1, x_3, \) and \( x_5 \), giving as solutions \[
\begin{bmatrix}
x_2 \\
x_5
\end{bmatrix}
\] or equivalently \( x_2 \begin{bmatrix}
-2 \\
1
\end{bmatrix} + x_5 \begin{bmatrix}
-3 \\
0
\end{bmatrix} \). So a basis for the solution space is the set of two vectors

\[
\begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
0 \\
2 \\
-4
\end{bmatrix}
\]
(b) What is the rank of \( A \)?

**ANSWER:**
You could look at the REF and read off the number of non-zero rows, 3. Or you could use the fact that the rank plus the dimension of the solution space must equal the number of columns: From (a) we know the dimension of the solution space is 2, and there are 5 columns, so the rank must be \( 5 - 2 = 3 \).

**Problem 7** (12 points)
For the vector space \( V = P_2 \) of polynomials with degree at most 2:

1. The set \( B_1 = \{1 + t, 1 + t^2, t^2\} \) is a basis for \( V \).
2. The set \( B_2 = \{t, 1 + t, 1 + t^2\} \) is also a basis for \( V \).

(a) Find the “change of basis” matrix (what the book denotes \( P_{B_1\leftarrow B_2} \)) that tells how coordinates with respect to \( B_2 \) change to become coordinates with respect to \( B_1 \).

**ANSWER:**
We find the coordinates of each vector in \( B_2 \) with respect to \( B_1 \), and write those as the columns of a matrix. For the first vector in \( B_2 \), \( t \), could either set up equations and solve them or just “think through it”. I see that I could get 1 as \((1 + t^2) - t^2\), and then \( t \) as \((1 + t) - 1 = (1 + t) - \[(1 + t^2) - t^2\] \).

So \( t = 1 \times (1 + t) + (-1) \times (1 + t^2) + 1 \times (t^2) \), i.e. the coordinates of \( t \) with respect to \( B_1 \) are 1, \(-1\), and 1 or as a column vector \[
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}
\].

Next we find the coordinates of \( 1 + t \) with respect to \( B_1 \): Since the first vector in \( B_1 \) is exactly \( 1 + t \), the coordinates are 1, 0, and 0 or \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\].

Similarly, \( 1 + t^2 \) is exactly the second vector in \( B_1 \), so its coordinates are 0, 1, or 0 or \[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\].

So the matrix we want is \( P_{B_1\leftarrow B_2} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \).

(b) Find the coordinates of \( 1 + t - t^2 \) with respect to \( B_2 \).

**ANSWER:**
Reasoning as before (You could instead set this up as a system of equations to be solved):
\[
1 = (1 + t) - t. \quad t^2 = (1 + t^2) - 1 = (1 + t^2) - [(1 + t) - t] = (1 + t^2) - (1 + t) + t. \quad \therefore 1 + t - t^2 = (1 + t) - t^2 = (1 + t) - [(1 + t^2) - (1 + t) + t] = -1 \times (t) + 2 \times (1 + t) - 1 \times (1 + t^2)
\]
and hence the coordinates as a vector are \[
\begin{bmatrix}
-1 \\
2 \\
-1
\end{bmatrix}
\].

(c) Find the coordinates of \( 1 + t - t^2 \) with respect to \( B_1 \).

**ANSWER:**
This time the calculations are simpler: \( 1 + t \) and \( t^2 \) are themselves basis vectors in \( B_1 \), so we can see \( 1 + t - t^2 \) as \( 1 \times (1 + t) + 0 \times (1 + t^2) - 1 \times (t^2) \), i.e. the coordinates as a vector are \[
\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}
\].
(d) As a check on your work, multiply the matrix you got in (a) on the left of the coordinate vector resulting from (b) and show that you get the vector corresponding to (c).

**ANSWER:**

If we multiply
\[
\begin{bmatrix}
1 & 1 & 0 \\
-1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-1 \\
2 \\
-1
\end{bmatrix}
\] we do get
\[
\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}.
\]

Problem 8  
(12 points)

Let \( A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \).

(a) Find the adjoint \( \text{adj}(A) \).

**ANSWER:**

Remember that the determinant of a \( 2 \times 2 \) matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is \( ad - bc \). We need to compute nine determinants, each being just that easy, and affix \( \pm \) signs. In the upper left of our \( 3 \times 3 \) matrix we put, with a plus sign, the determinant of the matrix we get if we delete the first row and column from our matrix, i.e. \( \text{det} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \). That is \( 2 \times 1 - 2 \times 1 = 0 \). For the entry in the \( i, j \) position we take the determinant of the \( 2 \times 2 \) matrix we would get by deleting the \( j \)th row and the \( i \)th column (don’t forget the transpose here, i.e. the swap of \( i \) and \( j \)), and use for a sign \(-1^{i+j}\). So for example in the middle of the top row, \( i = 1 \) and \( j = 2 \), we take the determinant of \( \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) and get \( 1 \times 1 - 0 \times 1 \), and \(-1^{1+2} = -1^3 = -1 \) so we use a minus sign. Filling in the calculations without the \( \pm \) signs we have
\[
\begin{bmatrix}
2 \times 1 - 2 \times 1 & 1 \times 1 - 0 \times 1 & 1 \times 2 - 0 \times 2 \\
2 \times 1 - 2 \times 0 & 2 \times 1 - 0 \times 0 & 2 \times 2 - 0 \times 2 \\
1 \times 2 - 0 \times 2 & 2 \times 1 - 1 \times 0 & 2 \times 2 - 1 \times 2
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
0 & 1 & 2 \\
2 & 2 & 4 \\
2 & 2 & 2
\end{bmatrix}.
\]

Now we put in the signs in their checkerboard pattern and get for the answer
\[
\begin{bmatrix}
0 & -1 & 2 \\
-2 & 2 & -4 \\
2 & -2 & 2
\end{bmatrix}.
\]

(b) Find the determinant \( \text{det}(A) \)

**ANSWER:**

We could of course just calculate \( \text{det}(A) \) directly, but we also know that \( A \times \text{adj}(A) \) should give us \( \text{det}(A) \times I_3 \), so we can both get the determinant and have a check on our answer to (a) if we just multiply \( A \) and our answer. We get
\[
\begin{bmatrix}
2 & 1 & 0 \\
2 & 2 & 2 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 2 \\
-2 & 2 & 4 \\
2 & -2 & 2
\end{bmatrix}
= \begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

which is a scalar matrix, i.e. a scalar times \( I_3 \) (validating our computation of the adjoint) where the scalar is \(-2\), so the determinant must be \(-2\).
(c) Find the inverse $A^{-1}$.

**ANSWER:**

Again there is a brute-force way to do this, but since $A \ adj(A) = det(A) \times I_3$, $A^{-1}$ has to be $\frac{1}{det(A)} \ adj(A)$ (so long as $det(A) \neq 0$, which is the case here but anyway would have to be true for $A^{-1}$ to exist). So $A^{-1} = \left( \frac{1}{-2} \right) \ adj(A) = \begin{bmatrix} 0 & \frac{1}{2} & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$.

**Problem 9**  (12 points) Prove that the only subspaces of $\mathbb{R}^2$ are

1. The subspace consisting of just $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
2. The subspace consisting of all multiples of some non-zero vector $\vec{v}$, and
3. The whole space $\mathbb{R}^2$.

**ANSWER:**

Let $V$ be some subspace of $\mathbb{R}^2$. Since $V$ is a subspace it must at least contain $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: If that is all it contains we are through since that is case (1). So we can assume there is something other than $\vec{0}$ in $V$, call it $\vec{u}$. Let $W$ be the set of all scalar multiples of $\vec{u}$. That makes $W$ the span of $\{\vec{u}\}$, which is a subspace of $\mathbb{R}^2$. Since $V$ is also a subspace (of $\mathbb{R}^2$), it is closed under scalar multiplication, so $W$ is contained in $V$. If $V$ consists only of $W$, we are again through since that is case (2).

If we are not yet through, then $V$ must contain some vector $\vec{v}$ that is not in $W$ and hence not a multiple of $\vec{u}$. Hence in the set $S = \{\vec{u}, \vec{v}\}$, neither vector is a linear combination (multiple) of preceding ones, so $S$ is a linearly independent set of two vectors in the space $\mathbb{R}^2$, which has dimension 2. We know a set of $n$ linearly independent vectors in a space of dimension $n$ must be a basis for that space, so $S$ is a basis for $\mathbb{R}^2$. But $S$ is contained in $V$, and $V$ has to be closed under linear combinations since it is a subspace, so the span of the vectors in $S$ must be contained in $V$. But since $S$ was a basis for $\mathbb{R}^2$, that span is all of $\mathbb{R}^2$. So $V$ contains all of $\mathbb{R}^2$. But $V$ is also contained in $\mathbb{R}^2$, so $V = \mathbb{R}^2$, which is case (3), and we are through.