Problem 1

For the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$:

(a) Find a basis for the solution space (null space) of $A$, the subspace of $\mathbb{R}^5$ consisting of solutions of $A\vec{x} = \vec{0}$.

**ANSWER:**

First we reduce $A$ to Reduced Row Echelon Form, getting $A_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. We see that the fifth column does not contain a leading entry, so the corresponding variable $x_5$ can be given an arbitrary value. From the first row we see that $x_1 + x_5 = 0$, i.e. $x_1$ must be $-x_5$. Similarly from the remaining rows we get $x_2 = x_5$, $x_3 = -2x_5$, and $x_4 = -2x_5$. So solutions must look like $\begin{bmatrix} -x_5 \\ x_5 \\ -2x_5 \\ -2x_5 \\ x_5 \end{bmatrix}$, i.e. the solutions are all multiples of $\begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$. Hence a basis for the solution space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) What is the dimension of the null space of $A$ (i.e. the nullity of $A$)?

**ANSWER:**

Since there was one vector in the basis, the dimension of the null space is 1.

Problem 2

For the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$ (the same matrix as in problem 1):

(a) What is the rank of $A$?

**ANSWER:**
We saw in the answer to problem 1 that the reduced row echelon form of $A_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Since there are four non-zero rows in that matrix, the rank of $A$ is 4.

(b) Find a basis for the row space of $A$.

**ANSWER:**

From (a) we know that the row space will be a four dimensional subspace of $\mathbb{R}^5$, so we know we need four 5-element row vectors. We could use the rows of $A$, or the rows of $A_R$.

Using the rows of $A$, one basis is $\{[1, 2, -2, 3, 1], [0, 1, 0, 0, -1], [0, 0, 1, 0, 2], [0, -1, 1, 1, 5]\}$.

(c) Find a basis for the column space of $A$ that consists of some columns of $A$.

**ANSWER:**

We can use the columns from $A_R$ that have leading entries to pick out columns from $A$.

The leading entries are in the first four columns, so we use $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

**Problem 3**

For each of the following functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, tell whether it is a linear transformation or not and give reasons for your answer:

(a) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2b \end{bmatrix}$.

**ANSWER:**

This is a linear transformation. You could check it directly from the definition, or you could write it as $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$. We know that multiplication by a matrix always gives a linear transformation.

(b) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2 \end{bmatrix}$.

**ANSWER:**

This is not a linear transformation. $L\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so this function does not take the zero vector to the zero vector, which any linear transformation must do.

(c) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a^2 \end{bmatrix}$.

**ANSWER:**

This is not a linear transformation. If we can find any instance where the requirements for a linear transformation fail, that would justify this claim, so there are many possible reasons to give. I note that $L\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, while $2L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. So this function does not satisfy $L(c\vec{v}) = cL(\vec{v})$. 
Problem 4

Let 
\[
A = \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}.
\]

(a) Find the characteristic polynomial of \( A \),

**ANSWER:**

\[
\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \lambda + 1 & 4 \\ 1 & \lambda - 2 \end{bmatrix}
\]

The characteristic polynomial of \( A \) is the determinant of that matrix, i.e. \((\lambda + 1)(\lambda - 2) - 4 \times 1 = \lambda^2 - \lambda - 6\).

(b) What are the eigenvalues of \( A \)?

**ANSWER:**

We can factor \( \lambda^2 - \lambda - 6 \) as \((\lambda - 3)(\lambda + 2)\), so the eigenvalues are \( \lambda = 3 \) and \( \lambda = -2 \).

(c) For each of the eigenvalues, describe all of the eigenvectors.

**ANSWER:**

We substitute each value of \( \lambda \) into the matrix \( \lambda I - A \) above and solve the corresponding homogeneous equations.

For \( \lambda = 3 \):

\[
\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

has solutions \( x_1 = -x_2 \), i.e. all multiples of \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), so the eigenvectors are all the non-zero multiples of \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

For \( \lambda = -2 \):

\[
\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

has solutions \( x_1 = x_2 \), i.e. all multiples of \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), so the eigenvectors are all the non-zero multiples of \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Problem 5

Assume \( L \) is a linear transformation from a vector space \( V \) to a vector space \( W \).

Prove that the range of \( L \) is a subspace of \( W \).

**ANSWER:**

The range, the set of all vectors \( \vec{w} \) in \( W \) such that \( \vec{w} = L(\vec{v}) \) for some \( \vec{v} \) in \( V \), is by definition a subset of \( W \). We need to show (a) it is not empty, (b) it is closed under addition, and (c) it is closed under multiplication by scalars.

(a) Since \( L(\vec{0}_V) = \vec{0}_W \) for any linear transformation from \( V \) to \( W \), we have that \( \vec{0}_W \) is \( L(\text{something in } V) \), so the range of \( L \) contains \( \vec{0}_W \), so the range is not empty.

(b) We need to show that the sum of any two vectors in the range produces a vector in the range. Suppose \( \vec{w}_1 \) and \( \vec{w}_2 \) are any two vectors in the range of \( L \). Since they are in the range, there must be vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) in \( V \) such that \( \vec{w}_1 = L(\vec{v}_1) \) and \( \vec{w}_2 = L(\vec{v}_2) \). But then \( \vec{w}_1 + \vec{w}_2 = L(\vec{v}_1) + L(\vec{v}_2) \), and since \( L \) is a linear transformation that must be \( L(\vec{v}_1 + \vec{v}_2) \), so \( \vec{v}_1 + \vec{v}_2 \) is a vector in \( V \) that \( L \) takes to \( \vec{w}_1 + \vec{w}_2 \), hence \( \vec{w}_1 + \vec{w}_2 \) is in the range of \( L \).
(c) We need to show that any scalar multiple of a vector in the range of $L$ is in the range of $L$. Let $\vec{w}$ be any vector in the range of $L$, and let $c$ be any scalar. Since $\vec{w}$ is in the range, $\vec{w} = L(\vec{v})$ for some $\vec{v} \in V$. Then $c\vec{w} = cL(\vec{v}) = L(c\vec{v})$ (since $L$ is a linear transformation), hence $c\vec{w}$ is “$L$ of something in $V$”, i.e. $c\vec{w}$ is in the range of $L$.

**Problem 6**

Let $L$ be the linear transformation from $P_2$ (the space of polynomials of degree at most two) to $P_2$ defined by $L(p(t)) = p'(t)$, the derivative of the polynomial function. Using the “standard” ordered basis $B = \{1, t, t^2\}$ (with the vectors in that order!):

(a) Find the matrix $A$ representing $L$ with respect to $B$ and $B$.

**ANSWER:**

We apply $L$ to (i.e. take the derivative of) each vector in $B$, and find the coordinate vector of the result with respect to $B$.

For the first vector $1$ in $B$, the derivative gives $0$ which is $0 \times 1 + 0 \times t + 0 \times t^2$, so the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. For the second vector $t$, the derivative is $1$, so the coordinate vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Lastly, the derivative of $t^2$ is $2t = 0 \times 1 + 2 \times 2 + 0 \times t^2$, and the coordinate vector of that is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Putting these together, the matrix is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) For the polynomial $p(t) = 3 - 2t + 2t^2$, what is the coordinate vector $[p(t)]_B$?

**ANSWER:**

Since $3 - 2t + 2t^2$ is already written as a linear combination of $1, t, t^2$, we can read off the coordinate vector $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

(c) What is the coordinate vector $[L(p(t))]_B$ for $L(p(t))$?

**ANSWER:**

Taking the derivative, $L(3 - 2t + 2t^2) = -2 + 4t$, so its coordinate vector is $\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$.

(d) Use the matrix from (a) and the vectors from (b) and (c) to show that the matrix “does the right thing”, i.e. that multiplying a coordinate vector by the matrix does give you the coordinates for the result of applying $L$.

**ANSWER:**

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$.
Problem 7
Suppose that \( L \) is a linear transformation from a vector space \( V \) to a vector space \( W \), and that the kernel of \( L \) contains only the zero vector of \( V \). Show that \( L \) must be \( 1 - 1 \).

**ANSWER:**
Assume that \( L \) is a linear transformation from \( V \) to \( W \) and that only \( \vec{0}_V \) is in the kernel of \( L \), i.e. \( \vec{0}_V \) is the only vector in \( V \) that \( L \) takes to \( \vec{0}_W \).

Then if \( L(\vec{u}) = L(\vec{v}) \), \( L(\vec{u}) - L(\vec{v}) = \vec{0}_W \), and since \( L \) is linear that tells us \( L(\vec{u} - \vec{v}) = \vec{0}_W \), i.e. \( \vec{u} - \vec{v} \) is in the kernel of \( L \). But the only vector in the kernel is \( \vec{0}_V \), so we must have \( \vec{u} - \vec{v} = \vec{0}_V \). Then \( \vec{u} = \vec{v} \). So we have shown that whenever \( L \) takes two vectors to the same result in \( W \), the two were really the same to begin with, i.e. \( L \) is \( 1 - 1 \).

Problem 8
For the vector space \( V = \mathbb{R}^3 \), with ordered bases \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \) and \( T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \):
Find the matrix \( P_{S \leftarrow T} \) for changing coordinates from \( T \) to \( S \).

**ANSWER:**
For each vector in \( T \), we find its coordinates with respect to \( S \). For the first vector, \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), we need to solve \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) for \( a \), \( b \), and \( c \). You can set that up as a system of equations, but we can also just see the answer: That last vector in \( S \) is exactly what we want, so \( a = b = 0 \) and \( c = 1 \), and the coordinate vector is \( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

Now for the second vector, \( \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \), solving to make \( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) give that vector: The vectors multiplied by \( a \) and by \( b \) have 0 in the third entry, so \( c \) must be 2 to make the third place work out. But that puts a 2 in the middle place: We fix that by making \( b = -2 \). So far that puts a 0 in the top position, so we let \( a = -1 \) and get the coordinates as \( \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \).

Moving to the third vector, in the same way we find \( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), so the coordinate vector is \( \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \).

We assemble these as the matrix, getting \( P_{S \leftarrow T} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \).
Problem 9

The set of vectors $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$, the vector space of three element column vectors.

Using the ordinary “dot product” as an inner product on $\mathbb{R}^3$:

(a) Use the Gram-Schmidt process starting with $B$ to find an orthogonal basis for $\mathbb{R}^3$, i.e. a basis where each pair of distinct vectors is orthogonal.

**ANSWER:**

To make the notation match both our textbook and my online description, I will give names to the three vectors making up $S$, $u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, and $u_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

We create new vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ that are orthogonal as follows. Start by letting

$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Now make $\vec{v}_2$ by starting with $\vec{u}_2$ and subtracting its projection onto $\vec{v}_1$: $\vec{v}_2 = \vec{u}_2 - \frac{(\vec{u}_2, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1$.

$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{u}_2 - \frac{12}{3} \vec{v}_1 = \vec{u}_2 - 4 \vec{v}_1$. But for now all we care about is orthogonality, not magnitude, so we could instead use 3 times that result, $3\vec{u}_2 - 4\vec{v}_1$, and not have to deal with fractions: (You did not need to do that, it just makes the arithmetic easier to follow!) We get $\vec{v}_2 = 3 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

(As a check, the dot product of $v_1$ and $v_2$ is now $(2 \times -2) + (1 \times 2) + (2 \times 1) = 0$, so these are indeed orthogonal!)

Now we construct $\vec{v}_3$ by starting with $\vec{u}_3$ and subtracting its projections onto each of $\vec{v}_1$ and $\vec{v}_2$, $\vec{v}_3 = \vec{u}_3 - \frac{(\vec{u}_3, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1 - \frac{(\vec{u}_3, \vec{v}_2)}{(\vec{v}_2, \vec{v}_2)} \vec{v}_2$. Computing those inner (dot) products: We already had $\vec{v}_1 \cdot \vec{v}_1 = 1$. $\vec{u}_3 \cdot \vec{v}_1 = -2 + 1 + 0 = -1$. $\vec{u}_3 \cdot \vec{v}_2 = 2 + 2 = 4$. $\vec{v}_2 \cdot \vec{v}_2 = 4 + 4 + 1 = 9$. So the formula above gives us $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{4}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. Again we can simplify things by using 9 times that vector for $\vec{v}_3$ to eliminate fractions, getting

$\vec{v}_3 = 9 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, and again we can check that this does give 0 for the dot product with either $\vec{v}_1$ or $\vec{v}_2$. Summarizing, our new, orthogonal, basis is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$.
(b) Continue from what you found in (a) to get a basis which is orthonormal, i.e. in addition to being orthogonal it now has the magnitude (norm, size) of each vector equal to 1.

**ANSWER:**
We multiply each of these vectors by \( 1/||\vec{v}|| \): Each has magnitude \( ||\vec{v}|| = \sqrt{9} = 3 \), so the resulting vectors are

\[
\begin{pmatrix}
\frac{-2}{3} \\
\frac{-2}{3}
\end{pmatrix},
\begin{pmatrix}
\frac{1}{3} \\
\frac{-2}{3}
\end{pmatrix},
\begin{pmatrix}
\frac{1}{3} \\
\frac{-2}{3}
\end{pmatrix},
\begin{pmatrix}
\frac{1}{3} \\
\frac{-2}{3}
\end{pmatrix}
\].

**Problem 10**
Suppose \( L \) is a linear transformation from \( V \) to \( W \). Prove:
If \( L \) is \( 1-1 \) and \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) is a linearly independent set in \( V \), then \( \{L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k)\} \) is a linearly independent set in \( W \).

**ANSWER:**
We are given that \( L \) is a linear transformation from \( V \) to \( W \) which is \( 1-1 \), so its kernel is just \( \vec{0}_V \). For the given linearly independent set \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) in \( V \), consider the vectors \( L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k) \) in \( W \) and suppose some linear combination \( a_1L(\vec{v}_1) + a_2L(\vec{v}_2) + \cdots + a_kL(\vec{v}_k) \) gives the zero vector \( \vec{0}_W \) in \( W \). Since \( L \) is linear we can rewrite that as \( L(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W \). But that says \( a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k \) is in the kernel of \( L \), so it must be \( \vec{0}_V \), i.e. \( a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V \), but the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) are linearly independent, so the coefficients \( a_1 = a_2 = \cdots = a_k = 0 \). Recapping, any linear combination of the vectors \( L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k) \) that gives \( \vec{0}_W \) must have all zero coefficients, so the vectors are linearly independent.