First Midterm Exam March 1, 2007 ANSWERS

Notice that some problems can be done in various ways, and I have given only one possible answer. Also remember that answers could be correct despite looking quite different from those given: In particular this applies to indefinite integrals, where the “+C” can take many forms. For example, \( \sin^{-1} \theta = -\cos^{-1} \theta + C \) a constant that could be included in +C.

**Problem 1**
Evaluate the integrals:

(a) \[ \int_{1}^{2} x \ln(x) \, dx \]

**Answer:**
We can do this using integration by parts. If we let \( u = \ln x \), then \( du = \frac{1}{x} \, dx \). Now \( dv \) has to be whatever is left to make up the integrand, so \( dv = x \, dx \), hence \( v = \frac{x^{2}}{2} \). Using the integration by parts formula \( \int u \, dv = uv - \int v \, du \), we have \( \int_{1}^{2} x \ln(x) \, dx = \left[ (\ln x) \left( \frac{x^{2}}{2} \right) \right]_{1}^{2} - \int_{1}^{2} \left( \frac{x^{2}}{2} \right) \left( \frac{1}{x} \right) \, dx \)
\[ = \left[ (\ln x) \left( \frac{x^{2}}{2} \right) \right]_{1}^{2} - \left[ \frac{x^{2}}{2} \right]_{1}^{2} = (2 \ln 2 - \frac{1}{2} \ln 1) - (2 - \frac{3}{4}) = 2 \ln 2 - \frac{3}{4}. \]

(b) \[ \int \cos^{3}(x) \, dx \]

**Answer:**
This odd power of the cosine can be integrated by keeping one copy of cosine to go with \( dx \) and replacing \( \cos^{2} x \) by \( 1 - \sin^{2} x \), then (if you want to be formal) substituting \( u = \sin x \) so \( du = \cos(x) \, dx \). We have \( \int \cos^{3}(x) \, dx = \int (1 - \sin^{2} x) \cos(x) \, dx = \int \cos(x) \, dx - \int \sin^{2}(x) \cos(x) \, dx = \sin(x) - \frac{1}{3} \sin^{3}(x) + C. \) (Note that this answer can be written in other ways that don’t look at all the same, using trig identities to produce an answer that differs by a constant which is swallowed in \( C \).)

**Problem 2**
Evaluate the integral \( \int \frac{5x - 3}{(x + 1)(x - 3)} \, dx \).

**Answer:**
This seems to call for a partial fraction rewriting of the integrand. We know we can write \( \frac{5x - 3}{(x + 1)(x - 3)} \, dx = \frac{A}{x + 1} + \frac{B}{x - 3} \) for some constants \( A \) and \( B \). Multiplying both sides by \((x + 1)(x - 3)\) we have \( 5x - 3 = A(x - 3) + B(x + 1) \). Expanding we get \( 5x - 3 = Ax - 3A + Bx + B \). Equating the \( x \) terms we have \( A + B = 5 \), and from the constants we have \(-3A + B = -3 \). You can solve these for \( A \) and \( B \) in several ways. One way: Multiply \( A + B = 5 \) by 3 to get \( 3A + 3B = 15 \). Add that equation to \(-3A + B = -3 \) and you get \( 4B = 12 \), so \( B = 3 \). Putting that in \( A + B = 5 \) gives \( A = 2 \).

So now we know \( \int \frac{5x - 3}{(x + 1)(x - 3)} \, dx = \int \frac{2}{x + 1} \, dx + \int \frac{3}{x - 3} \, dx \). Each of those integrals is of the form \( \int \frac{1}{u} \, du \), so the answer is \( 2 \ln |x + 1| + 3 \ln |x - 3| + C. \)
Problem 3
A parabola centered at (0, 0) and opening upwards goes through the point (-4, 1).
Find an equation for this curve. What are the coordinates of its focus? (Write out the equation and
the coordinates explicitly!)

ANSWER:
We know we can write the equation for a parabola that is in standard position and opens upward as
\( y = \frac{1}{4p} x^2 \), for some number \( p \). But since (-4, 1) is on the curve, \( 1 = \frac{1}{4p} (-4)^2 = \frac{3}{p} \). Hence \( p = 4 \), and
the equation for the parabola is \( y = \frac{1}{16} x^2 \).
We also know that a parabola in this position has its focus at \((0, p)\) on the \( y \)-axis, i.e. at \((0, 4)\).

Problem 4

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  1 & 3 \\
  2 & 2 \\
  3 & 3 \\
  4 & 1 \\
\end{array}
\]

A function \( f(x) \) obtained from real-world measurements takes on these values:

Estimate \( \int_1^4 f(x) \, dx \) using one of our numerical integration techniques, Simpson’s Rule
or the trapezoidal Rule: Be sure to specify which you are using!

ANSWER:
We have the interval \([1, 4]\) divided into \( n = 3 \) subintervals. Simpson’s rule only works for an even
number of subintervals so we have to use the trapezoidal rule. (Since most people think that is easier
to use, I suspect most people taking the test would have gone this way even without that argument!)
Our subinterval end points are \( x_0 = 1, \, x_1 = 2, \, x_2 = 3, \) and \( x_3 = 4. \) The corresponding function values
are \( f(1) = 3, \, f(2) = 2, \, f(3) = 3, \) and \( f(4) = 1. \) The length of each subinterval is \( \Delta x = 1. \) Using the
trapezoidal rule we want to calculate \( \frac{\Delta x}{2} (f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(3+4+6+1) = \frac{1}{2} \times 14 = 7. \)

Problem 5
Find parametric equations \( x = f(t) \) and \( y = g(t) \) describing motion along the hyperbola
\(-\frac{x^2}{16} + \frac{y^2}{4} = 1\),
such that the point \((x, y)\) is at \((0, -2)\) when \( t = 0 \) and it moves to the right as \( t \) increases.

ANSWER:
We recall that \( \sec t \) and \( \tan t \) satisfy the identity \( \sec^2 t = 1 + \tan^2 t \), or equivalently \( \sec^2 t - \tan^2 t = 1. \)
If we let \( \sec^2 t = \frac{y^2}{4} \) and \( \tan^2 t = \frac{x^2}{16} \) then the equation for the hyperbola will be satisfied. That means
we can use \( x = \pm 4 \tan t \) and \( y = \pm 2 \sec t \), where we still have to choose the signs.
Putting in \( t = 0 \) has to give us \( x = 0 \) and \( y = -2 \): Since \( \sec 0 = 1 \), this forces us to pick the - sign for
\( y, \, y = -2 \sec t. \) Now as \( t \) increases from 0, \( \tan t \) also increases. We want \( x \) to be increasing, so that
the point moves to the right, so we have to choose the + sign for \( x. \) Hence the parametric equations
are \( x = 4 \tan t \) and \( y = -2 \sec t. \)
Problem 6

For the curve \( r = 4 \sin 3\theta \), find the slope where \( \theta = \frac{\pi}{4} \). The (a) plot to the right shows roughly what this curve looks like: You must calculate the slope using derivatives to get credit.

**ANSWER:**

We use the formula \( \frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} \) with \( f(\theta) = r = 4 \sin 3\theta \). Then \( f'(\theta) = 12 \cos 3\theta \).

At the point where \( \theta = \frac{\pi}{4} \): \( \sin \theta = \cos \theta = \frac{\sqrt{2}}{2} \), \( f(\theta) = 4 \sin 3\theta = 4 \sin \frac{3\pi}{4} = 2\sqrt{2} \), and \( f'(\theta) = 12 \cos 3\theta = 12 \cos \frac{3\pi}{4} = -6\sqrt{2} \).

Putting those numbers into the formula we note that the \( \sin \theta \) and \( \cos \theta \) factors in the numerator and denominator are all \( \frac{\sqrt{2}}{2} \) and so they cancel out. That leaves us with \( \frac{dy}{dx} = \frac{f'(\theta) + f(\theta)}{f'(\theta) - f(\theta)} = \frac{-6\sqrt{2} + 2\sqrt{2}}{-6\sqrt{2} - 2\sqrt{2}} = -\frac{4}{-8} = \frac{1}{2} \).

Find the points where \( r = 1 + \cos \theta \) and \( r = 1 - \cos \theta \) intersect. You can use the plot at the right as a check of (b) your work but you must show how you calculate the specific coordinates of each intersection point: Just reading the intersection points from the plot will not receive any credit.

**ANSWER:**

First we try setting the two \( r \) values equal. We get the equation \( 1 + \cos \theta = 1 - \cos \theta \) or \( 2 \cos \theta = 0 \), so \( \cos \theta = 0 \). That occurs for \( \theta = \pm \frac{\pi}{2} \). We put that into each function and find they both give \( r = 1 - 0 = 1 \) so each curve passes through the point \( (1, \frac{\pi}{2}) \) and through the point \( (1, -\frac{\pi}{2}) \), confirming the two points of intersection the picture shows on the upper and lower \( y \)-axis.

Now it appears both curves go through the origin which we think of as \( (0, 0) \), but that does not work in either equation! But \( r = 1 + \cos \theta \) takes on the value \( 0 \) when \( \cos \theta = -1 \), e.g. when \( \theta = \pi \), and \( r = 1 - \cos \theta \) gives \( 0 \) when \( \cos \theta = 1 \), e.g. when \( \theta = 0 \), so the origin is on each curve, just for different \( \theta \) values. So the origin, whether labelled \( (0, 0) \) or \( (0, \pi) \), is an intersection point for the curves.
Problem 7

(a) Sketch the curve $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Be sure to show where it crosses the x-axis and/or the y-axis (write out the coordinates!), and where its foci are (write out the coordinates!). Your sketch will not be graded for drawing ability but should resemble the correct curve.

**ANSWER:**
Here is Maple’s plot of this ellipse. The curve crosses the x-axis where $y = 0$, so $x^2 = 9$ and $x = \pm 3$. Similarly the y-intercept is where $y = \pm 5$. In our usual notation $a$ is the larger of those sizes, 5, and $b$ is the smaller, 3, so the distance from the center to a focus is $\sqrt{25 - 9} = 4$. Hence the intercepts are $(\pm 3, 0)$ and $(0, \pm 5)$, and the foci are at $(0, \pm 4)$.

(b) The equation $4x^2 + 2\sqrt{3}xy + 2y^2 + 10\sqrt{3}x + 10y = 5$ describes a conic section. (An ellipse, parabola, or hyperbola, not a degenerate case such as a line or point.)

(i) Find an angle $\theta$ such that rotation of the coordinate system by $\theta$ would eliminate the $xy$ term. (You do not need to carry out the rotation!)

**ANSWER:**
This equation has the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where $A = 4$, $B = 2\sqrt{3}$, $C = 2$, and the rest don’t matter for this problem. We can rotate by any angle $\theta$ that satisfies $\cot 2\theta = \frac{A-C}{B} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$. If $\cot 2\theta = \frac{1}{\sqrt{3}}$, $\tan 2\theta = \sqrt{3}$. So one choice is $2\theta = \frac{\pi}{3}$, in which case $\theta = \frac{\pi}{6}$.

(ii) Which kind of curve (ellipse, parabola, or hyperbola) is this conic section?

**ANSWER:**
We can use the discriminant $B^2 - 4AC = (2\sqrt{3})^2 - 4 \times 4 \times 2 = 12 - 32 = -20$. Since that is negative, the curve is an ellipse.
Problem 8

(a) Evaluate the integral \( \int \frac{x^3 \, dx}{\sqrt{16 - x^2}} \).

**ANSWER:**
It is possible to do this with a “\( u \) substitution” that does not involve trigonometry, but the difference of squares suggests a trig substitution so I will do it that way. If we draw a triangle and label it as

\[
\begin{align*}
4 & \quad \sqrt{16 - x^2} \\
\theta & \quad x
\end{align*}
\]

we are led to \( x = 4 \cos \theta \) so \( dx = -4 \sin \theta \), and \( \sqrt{16 - x^2} = 4 \sin \theta \). Thus the integral becomes

\[
\int \frac{(4 \cos \theta)^3 \, (-4 \sin \theta) \, d\theta}{4 \sin \theta} = -\int 64 \cos^3 \theta \, d\theta.
\]

We separate \( \cos^3 \theta \, d\theta \) as \( (\cos^2 \theta) \cos \theta \, d\theta \) and use \( \cos^2 \theta = 1 - \sin^2 \theta \), and have

\[
-64 \int (1 - \sin^2 \theta) \cos \theta \, d\theta = -64 \int \cos \theta \, d\theta + 64 \int \sin^2 \theta \cos \theta \, d\theta.
\]

We use the substitution \( u = \sin \theta \) on the second integral and get

\[
-64 \sin \theta + \frac{64}{3} \sin^3 \theta + C.
\]

From the triangle we read \( \sin \theta = \frac{\sqrt{16 - x^2}}{4} \) and substituting that in gives

\[
-16 \sqrt{16 - x^2} + \frac{4}{3}(16 - x^2)^{\frac{3}{2}} + C.
\]

(b) One of the integrals \( \int_1^\infty \frac{dx}{\sqrt{x}} \) and \( \int_1^\infty \frac{dx}{x^3} \) converges, and the other does not. Evaluate the one that converges.

**ANSWER:**
We could try evaluating each to see which one converges. But each is of the form \( \int_1^\infty \frac{dx}{x^p} \), which we know converges only if \( p > 1 \), whereas we have to choose between \( p = \frac{1}{2} \) and \( p = 3 \). So we go with \( p = 3 \) and evaluate \( \int_1^\infty \frac{dx}{x^3} \).

To evaluate this improper integral we need to set it up as a limit, \( \lim_{b \to \infty} \int_1^b \frac{dx}{x^3} \). The integration (power rule) yields

\[
-\frac{1}{2}x^{-2}\bigg|_1^b = -\frac{1}{2b^2} - \frac{1}{2} = \frac{1}{2} \left( 1 - \frac{1}{b^2} \right).
\]

Taking the limit as \( b \to \infty \) we get \( \frac{1}{2} \) as the answer.