Instructions:

If you signed up for Computability Theory, do two E and two C problems.

If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Show that every countable ordinal has the same order type as a closed set of reals.

E2. Show that the set \( \{\langle n, m, p \rangle \mid n + m = p \} \) is not definable in the structure \((\mathbb{N}, \cdot)\).

E3. Gödel’s First Incompleteness Theorem tells us that there is a Π sentence \( \varphi \) such that \( \varphi \) is true in \((\mathbb{N}, +, \cdot, 0, 1)\), but is not provable in Peano arithmetic. Is there a Σ sentence with this property? If so, write it down, and if not, prove that no Σ sentence has this property.

Recall that a Π sentence is one of the form \( \forall x \psi \), where \( \psi \) is quantifier free, and a Σ formula is one of the form \( \exists x \psi \), where \( \psi \) is quantifier free.
C1. Prove that there exists $x$ such that

$$\forall y (x \in W_y \iff y \in W_x).$$

C2. Let $A$ be c.e. Prove that there is no total $A$-computable function $f$ such that for all $e$, if $W_e$ is finite then $W_e \subseteq \{0, \ldots, f(e)\}$.

C3. Show that there is a noncomputable c.e. set $A$ such that for any disjoint c.e. sets $U$ and $V$ with $A = U \cup V$, if $A$ is $U$-computable then $V$ is computable.

Note: c.e. is the same as r.e., computable is the same as recursive, $A$ is $U$-computable is the same as $A$ is Turing reducible to $U$ or $A \leq_T U$, and $W_e$ is the $e^{th}$ c.e. set in some standard enumeration.
S1. Let $[\mathbb{R}]^{<\omega}$ denote the set of all finite subsets of the reals and $[\mathbb{R}]^\omega$ denote the set of all countably infinite subsets of the reals. Prove that CH is equivalent to the following statement:

(P) There is a function $F : [\mathbb{R}]^{<\omega} \to [\mathbb{R}]^\omega$ such that for every $A \in [\mathbb{R}]^{<\omega}$, we have $a \in F(A \setminus \{a\})$ for all but at most one $a \in A$.

S2. Prove there exists a family $\mathcal{P}$ of perfect subtrees of $2^{<\omega}$ which when ordered by reverse inclusion is an $\omega_1$-Aronszajn tree.

Notation. An $\omega_1$-Aronszajn tree is a tree $\mathcal{P}$ of height $\omega_1$ such that $\mathcal{P}$ has no uncountable branches and such that $\mathcal{P}_\alpha$ (the $\alpha$th level of $\mathcal{P}$) is countable for each $\alpha$. A subtree $p \subseteq 2^{<\omega}$ is perfect iff every node in $p$ has incompatible nodes above it.

Hint. You can construct $\mathcal{P}$ inductively, with the root equal $2^{<\omega}$. You have to make sure that $\mathcal{P}$ doesn’t die at limit levels, so maintain the property that for any $\alpha < \beta$, $n < \omega$, and $p \in \mathcal{P}_\alpha$ there exists $q \in \mathcal{P}_\beta$ with $q \subseteq p$ and $q \cap 2^n = p \cap 2^n$.

S3. Prove that the following is consistent with $\neg CH$: There are cofinal $A_\gamma \subset \gamma$, for $\gamma$ a countable limit ordinal, such that each $A_\gamma$ has order type $\omega$ and such that whenever $A$ is an unbounded subset of $\omega_1$, there is a closed unbounded $C \subseteq \omega_1$ such that $A \cap A_\gamma$ is infinite for all $\gamma$ in $C$.

Hint. Use Cohen forcing, and use the $\gamma$th Cohen real to code $A_\gamma$. 

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E1. Using induction on $\alpha < \omega_1$ and also that any two open intervals are order isomorphic. This can also be proved without using the axiom of choice.

E2. Take any permutation $\sigma$ of the primes. It extends to an automorphism of $(\mathbb{N}, \cdot)$ using the fundamental theorem of arithmetic. But $<$ is definable from $+$ and $(\mathbb{N}, <)$ has no nontrivial automorphisms.

E3. Any $\Sigma$ sentence true in $\mathbb{N}$ is provable in Peano arithmetic. All one has to prove is that all quantifier free sentences true in $\mathbb{N}$ are provable in PA. Let $n$ stand for $1 + 1 + 1 + \cdots + 1$ $n$-times (or 0 if $n = 0$). By induction one can prove that

$$PA \vdash n + m = n + m$$
$$PA \vdash n \cdot m = nm$$

$n \neq m$ implies $PA \vdash n \neq m$.

This collection of sentences is known as Robinson’s Q. It follows that any model of PA is a model of Q, hence any quantifier free sentence true in $\mathbb{N}$ is true in all models of PA.

S1. Assume CH. Replace $\mathbb{R}$ by $\omega_1$, and let $F(A) = A \cup \bigcup A = A \cup \bigcup_{n \in A} A$. Assume (P) and not CH. Take $C \subseteq \mathbb{R}$ of size $\omega_1$. There exists $a \in \mathbb{R} \setminus \bigcup_{b \in C} F(\{b\})$. For each $b \in C$ we have $b \in F(\{a\})$, so $F(\{a\})$ is uncountable, contradicting (P).

S2. A key argument for constructing the limit levels $\mathcal{P}_\lambda$ is the fusion lemma: If $(p_n : n < \omega)$ are a descending sequence of perfect trees and $(k_n : n < \omega)$ is an increasing sequence such that for all $n$ we have

$p_{n+1} \cap 2^{k_n} = p_n \cap 2^{k_n}$ and

all nodes in $p_{n+1} \cap 2^{k_n}$ have at least two extensions in $p_{n+1} \cap 2^{k_{n+1}}$, then $\cap_{n<\omega} p_n$ is a perfect tree. Another property which should be in the construction is that any distinct $p, q$ at the same level should have no infinite branches in common (or equivalently $p \cap q$ is finite). $\mathcal{P}$ has no $\omega_1$ branch since there cannot be an $\omega_1$ descending chain in the power set of $2^{<\omega}$.

S3. In the ground model, $V$, choose a map $f_\gamma$ from $\omega$ onto $\gamma$ for each countable limit $\gamma$. Let $V[G]$ add Cohen reals, $\{r_\alpha : \alpha < \kappa \} \subseteq \omega^\omega$. To ensure $\neg CH$, let $\kappa \geq \omega_2$ (or, assume $V \models \neg CH$). Use $r_\gamma$ to construct $A_\gamma$; for example, let $A_\gamma = \{ \xi^\gamma_n : n \in \omega \}$, where $\xi^\gamma_0 = 0$, and $\xi^\gamma_{n+1}$ is $f_\gamma(r_\gamma(n))$ if this is greater than $\max(f_\gamma(n), \xi^\gamma_n)$, and $\max(f_\gamma(n), \xi^\gamma_n + 1)$ otherwise.
C1. Prove that there exists $x$ such that
\[ \forall y (x \in W_y \iff y \in W_x). \]

Proof: Define a computable function $f$ such that $W_{f(x)} = \{y \mid x \in W_y\}$, and apply the Fixed-Point Theorem to get an index $x_0$ with $W_{x_0} = W_{f(x_0)}$.

C2. Let $A$ be c.e. Prove that there is no total $A$-computable function $f$ such that for all $e$, if $W_e$ is finite then $W_e \subseteq \{0, \ldots, f(e)\}$.

Proof: “$W_e$ is finite” is a $\Sigma^0_2$-complete property, whereas for any $A$-computable function $f$, “$W_e \subseteq \{0, \ldots, f(e)\}$” is $\Pi^0_2$.

C3. Show that there is a noncomputable c.e. set $A$ such that for any disjoint c.e. sets $U$ and $V$, if $A = U \cup V$ and one of $U$ or $V$ computes $A$ then the other is computable.

Proof by priority argument: Try to show $V$ computable at $x$ after $U$ computes $A(x)$. If later $x$ enters $A$ (and thus may now enter $V$), then either $x$ enters $U$ (and thus not $V$), or $x$ enters only $V$ and so we can kill the reduction from $U$ to $A$ at $x$ (as $A(x)$ is no longer correctly computed by $U$).

(Comment: This property is called mitotic and is known to be equivalent to autoreducibility, but this is not relevant here.)