The structure of Weihrauch degrees - what we know and what we don’t know

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2017: The survey

And an update

What happened since? What are some interesting open questions?

Arno Pauly:
An update on Weihrauch complexity, and some open questions.
arXiv 2008.11168
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A very short overview

- Weihrauch reducibility compares multivalued functions between represented spaces.
- The induced degrees have a rich algebraic structure.
- Many mathematical theorems can be interpreted as multivalued functions, with the associated Weihrauch degrees measuring the computational content of the theorem.
- The algebraic operations have logic-like meanings regarding such theorems.
- Many concrete theorems have been classified via Weihrauch reducibility; and this classification is reminiscent of reverse mathematics and Brouwerian counterexamples.
- Various techniques have been developed to prove separation results.
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Represented spaces and computability

**Definition**
A *represented space* $X$ is a pair $(X, \delta_X)$ where $X$ is a set and $\delta_X : \subseteq 2^\mathbb{N} \to X$ a surjective partial function.

**Definition**
$F : \subseteq 2^\mathbb{N} \to 2^\mathbb{N}$ is a realizer of $f : \subseteq X \Rightarrow Y$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \text{dom}(f\delta_X)$.

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\begin{array}{ccc}
2^\mathbb{N} & \xrightarrow{F} & 2^\mathbb{N} \\
\downarrow{\delta_X} & & \downarrow{\delta_Y} \\
X & \xrightarrow{f} & Y
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**Definition**
$f : \subseteq X \Rightarrow Y$ is called computable (continuous), iff it has a computable (continuous) realizer.
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Weihrauch-reducibility

Definition
For \( f : \subseteq X \Rightarrow Y \), \( g : \subseteq V \Rightarrow W \) say

\[
f \leq_W g
\]

iff there are computable \( H, K : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \), such that \( H\langle \text{id}_{\mathbb{N}^\mathbb{N}}, GK \rangle \) is a realizer of \( f \) for every realizer \( G \) of \( g \). \( W \) denotes the Weihrauch degrees.
Weihrauch reducibility on Baire space

Proposition

For \( f, g : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \) we that \( f \leq_W g \) iff there are computable
\( H, K : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \) with \( K : \text{dom}(f) \to \text{dom}(g) \) such that
\( H(\langle p, q \rangle) \in f(p) \) for all \( q \in g(K(p)) \).
Most work on Weihrauch degrees is about classifying specific theorems.

Then there is work on creating a “scaffolding” of stuff like closed choice principles.

But only a few papers on the structure of the Weihrauch degrees.

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The Weihrauch lattice

Structures embeddable in the Weihrauch degrees

More algebraic operations

Special subclasses

Some side comments

The big open questions
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Theorem (Brattka & Gherardi; Pauly)

The Weihrauch degrees form a distributive lattice;

- with join $\sqcup$ induced by $(f \sqcup g) : \subseteq X + U \Rightarrow Y + U$,
  $(f \sqcup g)(0, x) = (0, f(x))$ and $(f \sqcup g)(1, y) = (1, g(y))$,

- and with meet $\sqcap$ induced by $(f \sqcap g) : \subseteq X \times U \Rightarrow Y + V$,
  $(f \sqcap g)(x, y) = (0 \times f(x)) \cup (1 \times g(y))$. 
Distributive lattice

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*The Weihrauch degrees form a distributive lattice;*

- with join \( \sqcup \) induced by 
  \[
  (f \sqcup g)(x) = \begin{cases} 
  (0, f(x)) & \text{if } x < 1 \\
  (1, g(y)) & \text{if } x = 1, y \in Y
  \end{cases}
  \]
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Distributive lattice
Special degrees

- The least element is 0, the trivially true principle without instances.
- With 1 we denote the degree of $\text{id}_{\mathbb{N}^\mathbb{N}}$ comprised of all computable problems with a computable instance.
- And $\emptyset$ is the top element (which is probably fake).
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Incompleteness

Theorem (Higuchi & Pauly)
No non-trivial suprema exist in the Weihrauch lattice, meaning either $\bigcup_{i \in \mathbb{N}} f_i$ does not exist, or there is some $N \in \mathbb{N}$ with $\bigcup_{i \in \mathbb{N}} f_i = \bigcup_{i \leq N} f_i$.

Theorem (Higuchi & Pauly)
Some non-trivial infima exist, others do not.

Corollary
$\mathfrak{W}$ and $\mathfrak{W}^{\text{op}}$ are not isomorphic.
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$\mathcal{W}$ and $\mathcal{W}^{op}$ are not isomorphic.
Heyting algebra?

**Question (Brattka & Gherardi)**

*Is the Weihrauch lattice a Brouwer algebra, i.e. does*

$$\inf_{\leq W} \{ h \mid g \leq_W f \sqcup h \}$$

*exist for all f, g?*

**Theorem (Higuchi & Pauly)**

*The Weihrauch lattice is neither a Brouwer nor a Heyting algebra.*
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The big open questions
Medvedev degrees

Definition (Medvedev reducibility)
For $A, B \subseteq \mathbb{N}^\mathbb{N}$, $A \leq_M B$ iff $\exists F : B \rightarrow A$, $F$ computable. Let $M$ denote the Medvedev degrees.

Theorem (Brattka & Gherardi)
$A \mapsto c_A$, where $c_A(p) = A$, is a meet-semilattice embedding of $M$ into $\mathcal{W}$.

Theorem (Higuchi & Pauly)
$A \mapsto d_A$, where $d_A : A \rightarrow \{0\}$, is a lattice embedding of $M^\text{op}$ into $\mathcal{W}$. In fact, it is an isomorphism between $M^\text{op}$ and \{ $f \in \mathcal{W} | 0 <_W f \leq_W 1$ \}.

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Is there a lattice-embedding of $M$ into $\mathcal{W}$?
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Many-one degrees

Definition (Many-one reductions)
For $A, B \subseteq \mathbb{N}$, let $A \leq_m B$ iff there is a computable $F : \mathbb{N} \to \mathbb{N}$ with $F^{-1}(B) = A$.

Theorem (Brattka & Pauly)
The many-one degrees embed into $\mathcal{M}$.

Proof.
Let $p, q \in \mathbb{N}^\mathbb{N}$ be Turing incompatible. Map $A \subseteq \mathbb{N}$ to $\chi_A^{p,q} : \mathbb{N} \to \{p, q\}$ where $(\chi_A^{p,q})^{-1}(p) = A$. 

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What really is “and”?

Definition
We call $f$ join-irreducible, if $f \leq_W g \sqcup h$ implies that $f \leq_W g$ or $f \leq_W h$.

Most “natural” Weihrauch degrees are join-irreducible.

Definition
Let $f \times g : X \times U \Rightarrow Y \times V$ be defined via $(y, v) \in (f \times g)(x, u)$ iff $y \in f(x)$ and $v \in g(v)$.

Proposition (Brattka)
$(\mathbb{W}, 0, 1, \sqcup, \times, *)$ is a Kleene-algebra.
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**Sequential composition**

**Definition**
Let \( f \star g = \sup_{\leq W} \{ F \circ G \mid F \leq_W f \land G \leq_W g \} \).

**Theorem (Brattka & Pauly)**
\( \star \) actually is a total operation on Weihrauch degrees.

**Theorem (Dzhafarov, Goh, Hirschfeldt, Patey & Pauly)**
\( \text{RT}_2^2 \leq_W \text{SRT}_2^2 \star \text{COH}, \) but \( \text{RT}_2^2 \) and \( \text{SRT}_2^2 \times \text{COH} \) are incomparable.
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Theorem (Brattka & Pauly)

The minimum \( \min_{\leq W} \{ h \mid f \leq W g \star h \} \) always exists (and is denoted by \( g \rightarrow f \)), but in general none of the following have to exist:

1. \( \inf_{\leq W} \{ h \mid f \leq W h \star g \} \)
2. \( \inf_{\leq W} \{ h \mid f \leq W g \times h \} \)

This means that the Weihrauch degrees are not a model of any of the usual substructural logics people have studied.
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Closure under composition

Definition (Neumann & Pauly)
An input for $f^\diamond$ is a description of an abstract register machine operating on represented spaces with computable functions and $f$ as operations, together with an input on which the register machine halts. The output is whatever the register machine outputs.

This is supposed to capture closure under composition.
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Characterizations

Proposition

\( f^* \) is the least Weihrauch degree above \( f \) satisfying \( 1 \leq_W f^* \) and \( f^* \times f^* \equiv_W f^* \).

Theorem (Westrick 2020)

\( f^\diamond \) is the least Weihrauch degree above \( f \) satisfying \( 1 \leq_W f^\diamond \) and \( f^\diamond \star f^\diamond \equiv_W f^\diamond \).

- Open since CCA 2015
- There is a constant function \( f \) and a multivalued function \( g \) such that \( f \leq_W g^\diamond \), but no fixed finite number of applications of \( g \) suffices
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Algebraic structure, summary

We have the following operations on Weihrauch degrees:

1. \( f \sqcap g \), returning either an answer to \( f \) or an answer to \( g \) (OR)
2. \( f \sqcup g \), letting us choose between \( f \) and \( g \) (AND)
3. \( f \times g \), letting us both \( f \) and \( g \) in parallel (AND)
4. \( f \star g \), letting us first use \( g \), then \( f \) (AND)
5. \( f \rightarrow g = \min\{h | g \leq_W f \star h\} \) (Implication)
6. \( f^*, f^\diamond \) letting us use \( f \) finitely many times, in parallel or consecutively (bang, bang)
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The big open questions
The idea

Sometimes, we can understand a Weihrauch degree by figuring out how it relates to “simple” Weihrauch degrees.

Definition (Dzhafarov, Solomon & Yokoyama)
Let the first-order part of a Weihrauch degree \( f \) be:

\[
1_f := \sup_{\leq_{\text{w}}} \{ g : \subseteq N^N \Rightarrow N \mid g \leq_{\text{w}} f \}
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Definition (Valenti, Goh & Pauly)
Fix a represented space \( X \). The deterministic part of a Weihrauch degree \( f \) is

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Proposition (Hoyrup)
There is an $f$ with $\text{Det}_{\mathbb{N}^N}(f) <_W \text{Det}_{\mathbb{R}}(f)$.

Proposition (de Brecht, Pauly & Schröder)
For overt choice $\text{VC}_Q : \subseteq \forall (Q) \Rightarrow Q$ it holds that $1(\text{VC}_Q) \equiv_W \text{Det}_{\mathbb{N}^N}(\text{VC}_Q) \equiv_W 1$, but $\text{VC}_Q$ is not computable.

Question (Valenti, Goh & Pauly)
Is there some $f$ with $\text{Det}_{\mathbb{N}}(f) <_W \text{Det}_{\mathbb{N}^N}(1f)$? (It always holds that $\text{Det}_{\mathbb{N}}(f) \equiv_W 1 \text{ Det}_{\mathbb{N}^N}(f)$)
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There are $f, g \leq_W \text{lim}$ with $f \times g \equiv_W \text{lim}$.

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There is a Weihrauch degree $f$ such that there is no $g$ with $g \star g \equiv_W f$. 
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- Clearly $\sqcup$, $\sqcap$, $\emptyset$, 0 are definable just by $\leq_w$
- Are $\times$ or 1 definable from other operations? What about $\hat{}$?
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The Weihrauch degrees are a distributive lattice.

Every countable distributive lattice embeds into the Weihrauch degrees (via the Medvedev degrees).

Thus, any universally quantified statement using $\sqcup$ and $\sqcap$ is either provable from the axioms of distributive lattices or false in $\mathbb{W}$.

Can we extend this to additional operations?

A list of known axioms and non-axioms is available in “On the algebraic structure of Weihrauch degrees”, LMCS 2018.
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If we relativize Weihrauch reducibility relative to an arbitrary oracle, we get continuous Weihrauch reducibility.

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