Coideals and the Local Ramsey Property
- Preliminary -

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Ramsey’s theorem 1929

**Notation:** \( A^{[n]} = \{ B \subseteq A : |B| = n \} \), \( A^{[<\infty]} = \bigcup_n A^{[n]} \)

\[
A^{[\infty]} = \{ B \subseteq A : |B| = \infty \}
\]

**Infinite version:** For every finite coloring of \( \mathbb{N}^{[2]} \) and every \( A \in \mathbb{N}^{[\infty]} \) there is \( B \in A^{[\infty]} \) such that \( B^{[2]} \) is monochromatic.
Ramsey’s theorem 1929

Notation: \( A^{[n]} = \{ B \subseteq A : |B| = n \} \), \( A^{[<\infty]} = \bigcup_n A^{[n]} \)

\[ A^{[\infty]} = \{ B \subseteq A : |B| = \infty \} \]

Generalized infinite version: Given an integer \( n > 0 \), for every finite coloring of \( \mathbb{N}^{[n]} \) and every \( A \in \mathbb{N}^{[\infty]} \) there is \( B \in A^{[\infty]} \) such that \( B^{[n]} \) is monochromatic.
Ramsey property

Question: Given $X \subseteq \mathbb{N}^{[\infty]}$, is there $A \in \mathbb{N}^{[\infty]}$ such that $A^{[\infty]} \subseteq X$ or $A^{[\infty]} \cap X = \emptyset$?

Answer: Not in general.

Example: For $A, B \in \mathbb{N}^{[\infty]}$, $A \sim B$ iff $|A \triangle B| < \infty$

(AC) Pick an element $B_x$ of each class $x \in \mathbb{N}^{[\infty]} / \sim$.

Let $cl(A)$ denote the class of $A$ and define

$$X = \{ A \in \mathbb{N}^{[\infty]} : |A \triangle B_{cl(A)}| \text{ is even} \}$$
Metric Topology on $\mathbb{N}[\infty]$

Identify each $A \in \mathbb{N}[\infty]$ with the increasing sequence $\{A(j)\}_j$ of its elements. Define the metric $d$ on $\mathbb{N}[\infty]$ by:

$$d(A, B) = \begin{cases} 
0 & \text{if } A = B \\
\frac{1}{n+1} & \text{if } n = \min\{j : A(j) \neq B(j)\}
\end{cases}$$

Basic open sets: $[a] = \{B \in \mathbb{N}[\infty] : a \sqsupseteq B\}$, where $a \in \mathbb{N}[<\infty]$. 
A set $\mathcal{X} \subseteq \mathbb{N}^\omega$ is said to be **Ramsey** if for every $A \in \mathbb{N}^\omega$ there exists $B \in A^\omega$ such that $B^\omega \subseteq \mathcal{X}$ or $B^\omega \subseteq \mathcal{X}^c$.

**EXAMPLES:**
(1) The set $\mathcal{X} = \{B \in \mathbb{N}^\omega : \min B = 8\}$ is Ramsey.
(2) Let $a \subset \mathbb{N}$ be a finite subset. The basic metric set $\mathcal{X} = [a] = \{B \in \mathbb{N}^\omega : a \sqsubseteq B\}$ is Ramsey.

**NOTE:** These are open sets. Notice that $\mathcal{X}^c$ (which is closed) is also Ramsey.
The Ramsey Property

- Clopen sets are Ramsey (Nash-Williams, 1965). Recall that a set is clopen if it is both closed and open.
- Open sets are Ramsey (Galvin, 1968)
- Borel sets are Ramsey (Galvin and Prikry, 1973). A set is said to be Borel if it is an element of the $\sigma$-algebra generated by the collection of all open sets.

NOTE: Silver (1972) proved that analytic sets (continuous images of Borel sets) are Ramsey, but using metamathematical techniques.
The Ramsey Property

Ellentuck’s topology on \( \mathbb{N}^{[\infty]} \):

\[ [a, A] = \{ B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A \},\]

where \( A \in \mathbb{N}^{[\infty]} \) and \( a \subset A \) is finite.

A set \( X \subseteq \mathbb{N}^{[\infty]} \) is said to be completely Ramsey if for every nonempty \( [a, A] \) there is \( B \in [a, A] \) such that \( [a, B] \subseteq X \) or \( [a, B] \cap X = \emptyset \). \( X \) is said to be completely Ramsey null if for every nonempty \( [a, A] \) there is \( B \in [a, A] \) such that \( [a, B] \cap X = \emptyset \).
The Ramsey Property

**Theorem:** (Ellentuck, 1974) Let $X \subseteq \mathbb{N}^{[\infty]}$ be given. Then,

1. $X$ is completely Ramsey if and only if $X$ has the Baire property in Ellentuck’s topology.
2. $X$ is completely Ramsey null if and only if $X$ is meager in Ellentuck’s topology.
Coideals and the Local Ramsey Property

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1. Local Ramsey Property
Local Ramsey Property

• A family $H \subseteq \mathbb{N}^{[\infty]}$ is a coideal if it satisfies the following:
  (i) $A \subseteq B$ and $A \in H$ implies $B \in H$;
  (ii) $A \cup B \in H$ implies $A \in H$ or $B \in H$.

• $[a, A] = \{B \in \mathbb{N}^{[\infty]} : B \subseteq A$ and $a \subseteq B\}$

• (Mathias) Let $H \subseteq \mathbb{N}^{[\infty]}$ be a coideal. $X \subseteq \mathbb{N}^{[\infty]}$ is $H$-Ramsey if for every non empty $[a, A]$ with $A \in H$ there exists $B \in [a, A] \cap H$ such that $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$. $X$ is $H$-Ramsey null if for every non empty $[a, A]$ with $A \in H$ there exists $B \in [a, A] \cap H$ such that $[a, B] \cap X = \emptyset$. 
Local Ramsey Property

- A coideal $H$ is **selective** if and only if every decreasing sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ in $H$ has a *diagonalization* in $H$: i.e., there is $B \in H$ such that $B/n \subseteq A_n$ for all $n \in B$.

- $H$ is **semiselective** if for every sequence $\{D_n\}_n$ of dense open subsets in $(H, \subseteq^*)$, the family of its diagonalizations is dense: i.e., for every $A \in H$ there is $B \in H$ such that $B \subseteq^* A$ and $B/n \in D_n$ for all $n \in B$. 
Local Ramsey Property

• (Ellentuck, 1974). Let $X \subseteq \mathbb{N}^{[\infty]}$ be given.
  (i) $X$ is Ramsey if and only if $X$ has the Baire property.
  (ii) $X$ is Ramsey null if and only if $X$ is meager.

• (Mathias, 1977). Let $X \subseteq \mathbb{N}^{[\infty]}$ and let $H$ be a selective coideal.
  (i) $X$ is $H$-Ramsey if and only if $X$ has the $H$-Baire property.
  (ii) $X$ is $H$-Ramsey null if and only if $X$ is $H$-meager.

• (Farah, 1997). Let $H$ be a coideal. The following are equivalent:
  (i) $H$ is semiselective.
  (ii) The $H$-Ramsey subsets of $\mathbb{N}^{[\infty]}$ are exactly those sets having the $H$- Baire property ...
2. Local Ramsey property in terms of games

*Ideal games and Ramsey sets,*
jointly with Carlos Di Prisco and Carlos Uzcátegui.
Infinite game $G_H(a, A, X)$ \hspace{1cm} (Kastanas, Matet)

Fix a coideal $H \subseteq \mathbb{N}^{[\infty]}$, $X \subseteq \mathbb{N}^{[\infty]}$, $A \in H$ and $a \in \mathbb{N}^{[<\infty]}$

I: \hspace{0.5cm} A_0 \hspace{0.5cm} A_1 \hspace{0.5cm} \ldots \hspace{0.5cm} A_k \hspace{0.5cm} \ldots \\

II: \hspace{0.5cm} (n_0, B_0) \hspace{0.5cm} (n_1, B_1) \hspace{0.5cm} \ldots \hspace{0.5cm} (n_k, B_k) \hspace{0.5cm} \ldots \\

$A_0 \in H \cap [a, A]$; $A_k, B_k \in H$; $n_k \in A_k$, $B_k \subseteq A_k/n_k$; $A_{k+1} \subseteq B_k$

Player I wins if and only if $a \cup \{n_0, n_1, n_2 \ldots \} \in X$
Local Ramsey property in terms of games

• (Kastanas, 1983). $X$ is Ramsey if and only if for every $A \in \mathbb{N}^{[\infty]}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathbb{N}^{[\infty]}}(a, A, X)$ is determined.

• (Matet, 1993). Let $H$ be a selective coideal. $X$ is $H$-Ramsey if and only if for every $A \in H$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{H}(a, A, X)$ is determined.
Semiselectivity?

• (DP – M – U) Let H be a coideal. The following are equivalent:
  (i) H is semiselective.
  (ii) For every $X \subseteq \mathbb{N}[\infty]$, X is H-Ramsey if and only if for every $A \in H$ and $a \in \mathbb{N}[<\infty]$ the game $G_H(a,A,X)$ is determined.
REMARK

• The phenomena described so far have analogs in other contexts: structures where *a Ramsey property* can be defined and characterized in *Ellentuck-like* terms are known as *Topological Ramsey Spaces*. *(Will define formally later).*

• We gave an *abstract approach to the local Ramsey property* within the framework of topological Ramsey space. *(Will introduce this later).*
3. Local Ramsey “Theories” of Block Sequences

*Ramsey sets of block sequences of vectors*, jointly with Daniel Calderon and Carlos Di Prisco
(2021. Submitted)
Block sequences

- $\text{FIN}^{[\infty]}$ = block sequences of non empty finite sets
- $\text{FIN}_k^{[\infty]}$ = block sequences of “vectors” $p : \mathbb{N} \to \{0, 1, \ldots, k\}$ such that $\text{supp}(p) = \{n : p(n) \neq 0\}$ is finite and $k \in \text{range}(p)$

$\text{FIN}^{[\infty]}$ and $\text{FIN}_k^{[\infty]}$ are topological Ramsey spaces
(by Milliken and Todorcevic, respectively)

So our abstract local Ramsey theory applies. Yet...
Comparing local Ramsey theories for block sequences

• (Blass) An ultrafilter $U$ on $\text{FIN}$ is an **ordered-union ultrafilter** if it has a basis of sets of the form $FU(A)$ where $A \in \text{FIN}[\infty]$.

$U$ is said to be **stable** if for every sequence $\{D_n\}_n \subseteq \text{FIN}[\infty]$ such that $FU(D_n) \in U$ for every $n$, there is $E \in \text{FIN}[\infty]$ such that $FU(E) \in U$ and for every $n$ $E \leq^* D_n$. 

Comparing local Ramsey theories for block sequences

• (Eisworth) A family $H \subseteq \text{FIN}^{[\infty]}$ is **Matet-adequate** if
  
  (1) $H$ is closed under finite changes,

  (2) For all $A, B \in \text{FIN}^{[\infty]}$, if $A \in H$ and $A \leq B$ then $B \in H$.

  (3) $(H, \leq^*)$ is $\sigma$-closed,

  (4) If $A \in H$ and $\text{FU}(A)$ is partitioned into 2 pieces then there is $B \leq A$ in $H$ so that $\text{FU}(B)$ is included in a single piece of the partition (this is called the Hindman property).
Comparing local Ramsey theories for block sequences

• Stable ordered-union ultrafilter on FIN vs selective ultrafilter on FIN$^{[∞]}$
• Mate adequate family on FIN$^{[∞]}$ vs selective coideal on FIN$^{[∞]}$
Comparing local Ramsey theories for block sequences

• (Blass) For any ordered union ultrafilter $U$ on $\text{FIN}$, the following are equivalent.

1. $U$ is stable.
2. $U^{\infty} = \{A \in \text{FIN}^{[\infty]} : FU(A) \in U\}$ is selective
3. $U$ has the Ramsey property for pairs
Comparing local Ramsey theories for block sequences

(C – DP – M)

• The generalization of Blass’ result to \( \text{FIN}_k \) holds.

• Matet-adequate families and selective coideals on \( \text{FIN}^{[\infty]} \) coincide. The corresponding generalizations to \( \text{FIN}_k^{[\infty]} \) also coincide.
4. Abstract Approach to the Local Ramsey Property

Topological Ramsey spaces

- $(R, \leq, r)$
  
  $r : \mathbb{N} \times R \to AR$; $r(n, A)$ is the n-th approximation of A

For $a$ in AR and $A$ in R, $[a, A] = \{B \in R : B \leq A \text{ and } r(n, B) = a\}$;
(Use to define Ramsey set like in $\mathbb{N}^{[\infty]}$)

Todorcevic introduces axioms A1, A2, A3 and A4 for a structure $(R, \leq, r)$.

- A1, A2 permit to understand $R$ as a metric subspace of the Polish space $AR^\mathbb{N}$. They also make the family of sets $[a, A]$ a base for another topology on R (Ellentuck-like).
- A3 makes $R$ a closed subset of $AR^\mathbb{N}$.
- A4 says that $(R, \leq, r)$ satisfies a “pigeon hole principle”.
Topological Ramsey spaces

• (R, \leq, r) is a topological Ramsey space if Baire sets and Ramsey sets coincide (i.e., “Ellentuck’s theorem” holds).

(Todorcevic) If (R, \leq, r) satisfies A1 – A4, then it is a Topological Ramsey space.
Abstract coideals

• Given \((R, \leq, r)\) satisfying A1 – A4, a subset \(H \subseteq R\) is a coideal if:

  (a) \(A \in H\) and \(A \leq B\) implies \(B \in H\).

  (b) \(H\) satisfies a local version of A3.

  (c) \(H\) satisfies a local version of the pigeon hole principle A4.
Abstract coideals

• Almost reduction:
For $A, B \in R$, write $A \leq^* B$ if there exists an approximation $a$ such that $\emptyset \neq [a, A] \subseteq [a, B]$

With these definitions, the notions of $H$-Ramsey set, $H$-Baire set, dense open in $(H, \leq^*)$, selective coideal and semiselective coideal can be lifted to the framework of the topological Ramsey space $(R, \leq, r)$.
Local Ramsey property, captured abstractly.

(DP – M – N) Given \((R, \leq, r)\) satisfying \(A1 - A4\), if \(H \subseteq R\) is a coideal, then the following statements are equivalent:

(1) \(H\) is a semiselective.

(2) \(X \subseteq R\) is \(H\)-Ramsey iff \(X\) is \(H\)-Baire.

(3) \(X \subseteq R\) is \(H\)-Ramsey null iff \(X\) is \(H\)-Meager.
**REMARK**

An *ultrafilter* is a maximal filter on \((R, \leq)\) satisfying local versions of \(A3\) and \(A4\).

If we don’t assume local versions of \(A3\) and \(A4\), then...

- (Trujillo) It is possible to show the existence of an ultrafilter that is selective but not Ramsey!

(We don’t want that!) We want: *Selective* \(\rightarrow\) *Semiselective* \(\rightarrow\) *Ramsey*.

... And we get it if we add \(A3\) and \(A4\).
Interesting consequences

\[(DP - M - N)\]

- Forcing with \((H, \leq^*)\) adds no new elements \(R\), and if \(U\) is the \((H,\leq^*)\)-generic filter over some ground model \(V\), then \(U\) is a selective ultrafilter in \(V[U]\).
- If there exists a super compact cardinal, then every selective ultrafilter \(U \subseteq R\) is \((R,\leq^*)\)-generic over \(L(\mathbb{R})\).
- If there exists a super compact cardinal and \(H \subseteq R\) is a semiselective coideal, then all definable subsets of \(R\) are \(H\)-Ramsey.
Next?

• Abstract infinite games. Play the game in your favorite topological Ramsey space.


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Thank you all!