Regularity of fractional maximal functions

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joint work with João Pedro Ramos and Olli Saari (Universität Bonn)
Maximal functions

The centered Hardy–Littlewood maximal function is

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and its non-centered version, denoted by \( \tilde{M} \) is

\[ \tilde{M}f(x) := \sup_{B(z, r) \ni x} \frac{1}{|B(z, r)|} \int_{B(z, r)} |f(y)| \, dy. \]
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\]
Trivially,
\[
\|Mf\|_\infty \lesssim \|f\|_\infty,
\]
and the classical Hardy–Littlewood inequality is the weak \((1, 1)\) bound
\[
|\{ x \in \mathbb{R}^d : |Mf(x)| > \lambda \}| \leq \frac{C}{\lambda} \|f\|_1.
\]
What about the regularity of these operators?

Kinnunen (1997): for \( 1 < p \leq \infty \), if \( f \in W^{1,p} \) then \( Mf \in W^{1,p} \) and
\[
\|Mf\|_p + \|\nabla Mf\|_p \lesssim \|f\|_p + \|\nabla f\|_p
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and moreover,
\[
|\nabla Mf(x)| \leq M(|\nabla f|)(x) \quad \text{a.e. } x \in \mathbb{R}^d.
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**Idea:**
- $M$ is sublinear: $M(f + g) \leq M(f) + M(g)$. 
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- $M$ commutes with translations: $Mf(x + h) = Mf_h(x), \quad f_h := f(x + h)$.
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  |D_h Mf(x)| = \left| \frac{Mf(x + h) - Mf(x)}{|h|} \right| \leq \left| \frac{M(f_h - f)(x)}{|h|} \right| = |M(D_h f)(x)|
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  so

  $$\|D_h Mf\|_p \leq \|M(D_h f)\|_p \lesssim \|D_h f\|_p.$$
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The \(W^{1,1}\) problem.
Tanaka (2002): for the non-centered maximal function $\tilde{M}$ and $d = 1$, 

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refined later by Aldaz and Pérez-Lázaro (2006);

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Kurka (2015): for the centered $M$,

$$\text{Var}(Mf) \leq 240000 \text{Var}(f).$$
Endpoint Sobolev regularity

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- In higher dimensions, only known for $\tilde{M}$ and radial functions, Luiro (2017):
  \[ \| \nabla \tilde{M}f \|_{L^1(\mathbb{R}^d)} \lesssim \| \nabla f \|_{L^1(\mathbb{R}^d)}. \]
Fractional counterparts

For $0 < \beta < d$, define the fractional maximal function

$$M_\beta f(x) := \sup_{r > 0} \frac{r^\beta}{r^d} \int_{B(x,r)} |f| = \sup_{r > 0} \frac{1}{r^{d-\beta}} \int_{|y| \leq r} |f(x - y)| \, dy$$

and the fractional integrals as

$$I_\beta f(x) := c \int_{\mathbb{R}^d} \frac{f(x - y)}{|y|^{d-\beta}} \, dy, \quad \widehat{I_\beta f}(\xi) = |\xi|^{-\beta} \widehat{f}(\xi).$$
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Relation is

$$M_\beta f(x) \leq I_\beta |f|(x)$$

and

$$\|I_\beta f\|_q \lesssim \|M_\beta f\|_q \lesssim \|f\|_p$$

whenever

$$\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}, \quad 1 < p < d/\beta.$$
Moreover, for $p = 1$, there is the corresponding weak-type counterpart:

$$\left\{|x \in \mathbb{R}^d : |M_\beta f(x)| > \lambda\right\|^\frac{d-\beta}{d} \leq \frac{C}{\lambda} \|f\|_1.$$
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Again, from Kinnunen’s result, one has

$$| \nabla M_{\beta} f(x)| \lesssim M_{\beta} (|\nabla f|)(x),$$

which implies

$$\| \nabla M_{\beta} f \|_q \lesssim \| \nabla f \|_p$$

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But what about the endpoint case $p = 1$?
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But what about the endpoint case $p = 1$?

$\beta > 1$ holds as a consequence of a smoothing effect for $M_\beta$. 

If $f \in L^p(\mathbb{R}^d)$ with $1 < p < \infty$,

$$D_i l_1 f(x) = -R_i f(x), \quad l_1 f(x) := c \int_{\mathbb{R}^d} \frac{f(x - y)}{|y|^{d-1}} \, dy$$

where

$$R_i f(x) = \lim_{\varepsilon \to 0} c \int_{|y| > \varepsilon} \frac{y_i}{|y|^{d+1}} f(x - y) \, dy$$

are the Riesz transforms.
Fractional integral: a smoothing property

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What happens for the fractional maximal function?
Fractional HL: the smoothing effect

Kinnunen–Saksman (2003): \( f \in L^p(\mathbb{R}^d) \) with \( 1 < p < d \) and \( 1 \leq \beta < d/p \)

\[
|\nabla M_\beta f(x)| \lesssim M_{\beta-1} f(x) \quad \text{a.e. } x \in \mathbb{R}^d.
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As a consequence

$$\|\nabla M_\beta f\|_q \lesssim \|f\|_p \quad \text{for } \frac{1}{q} = \frac{1}{p} - \frac{\beta - 1}{d}, \quad 1 < p \leq q < \infty.$$
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For \( \beta \geq 1 \), \( M_\beta : L^p \rightarrow \dot{W}^{1,q} \) i.e.,

\[ f \text{ rough function, but } M_\beta f \text{ is smooth of order } 1 \text{ for } \beta \geq 1 \]
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$f$ rough function, but $M_\beta f$ is smooth of order 1 for $\beta \geq 1$

The $W^{1,1}$ problem for $1 \leq \beta < d$ then follows by Gagliardo–Nirenberg–Sobolev:

$$\|\nabla M_\beta f\|_{L^{d-\beta} (\mathbb{R}^d)} \lesssim \|M_{\beta-1} f\|_{L^{d-\beta} (\mathbb{R}^d)} \lesssim \|f\|_{L^{d-1} (\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$
**Question:** Does this smoothing effect persist when taking more singular averages?
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S_\beta f(x) := \sup_{r>0} r^\beta |f * d\sigma_r(x)| = \sup_{r>0} r^\beta \left| \int_{S^{d-1}} f(x - ry) d\sigma(y) \right|
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\]

For \( \beta = 0 \), the best one can do is

\[
\|Sf\|_p \leq C\|f\|_p \quad \text{for} \quad \frac{d}{d - 1} < p \leq \infty.
\]

- Stein (1976) for \( d \geq 3 \)
- Bourgain (1986) for \( d = 2 \).
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For \( \beta > 0 \), Schlag (1997), Schlag and Sogge (1997) and S. Lee (2003):

\[ \| S_\beta f \|_{L^q} \lesssim \| f \|_{L^p} \quad \text{for} \quad \frac{1}{q} = \frac{1}{p} - \frac{\beta}{d} \]

if \( d \geq 2, p > d/(d - 1) \) and \( 0 \leq \beta < \tilde{\beta}(p) \), where

\[ \tilde{\beta}(p) := \begin{cases} \frac{d}{d-1} - \frac{2d}{p(d-1)} & \text{if} \quad \frac{d}{d-1} < p \leq \frac{d^2+1}{d(d-1)} \\ \frac{d-1}{p} & \text{if} \quad \frac{d^2+1}{d(d-1)} < p \leq \infty. \end{cases} \]
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Kinnunen’s general argument (sublinear + commute with translations) also yields

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|\nabla S_\beta f(x)| \leq S_\beta(\nabla f)(x),
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Kinnunen’s general argument (sublinear + commute with translations) also yields

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|\nabla S_\beta f(x)| \leq S_\beta (\nabla f)(x),
\]

but the Kinnunen–Saksman argument is more attached to the geometry of \( M \) and does not yield the corresponding smoothing effect

\[
|\nabla S_\beta f(x)| \preceq S_{\beta-1} f(x).
\]
The smoothing effect

**Theorem (B.–Ramos–Saari, 2018)**

Let $d \geq 5$, $d / (d - 2) < p \leq q < \infty$ and

$$
\beta(p) := \begin{cases} 
\frac{d^2 - 2d - 1}{d - 1} - \frac{2d}{p(d-1)} & \text{if } \frac{d}{d-2} < p \leq \frac{d^2 + 1}{d^2 - 2d - 1} \\
\frac{d-1}{p} & \text{if } \frac{d^2 + 1}{d^2 - 2d - 1} < p \leq d - 1.
\end{cases}
$$

Assume that

$$
\frac{1}{q} = \frac{1}{p} - \frac{\beta - 1}{d}, \quad 1 \leq \beta < \beta(p).
$$

Then, for any $f \in L^p$, $S_\beta f$ is weakly differentiable and

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\| \nabla S_\beta f \|_{L^q} \lesssim \| f \|_{L^p}.
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Then, for any \( f \in L^p, \) \( S_\beta f \) is weakly differentiable and

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\| \nabla S_\beta f \|_{L^q} \lesssim \| f \|_{L^p}.
\]

\( f \) rough function, but \( S_\beta f \) is smooth of order 1 for \( \beta \geq 1 \).
The Kinnunen–Saksman approach

Need to obtain an upper bound for

\[ D_h M_\beta f(x) := \frac{M_\beta f(x + h) - M_\beta f(x)}{|h|}. \]
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\[ M_\beta f(x + h) = r_h^{\beta-d} \int_{B(x+h,r_h)} |f| . \]
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Let $r_h$ be a good radius for $M_\beta f(x + h) = r_h^{\beta - d} \int_{B(x+h,r_h)} |f|$. 

![Diagram showing the relation between $x$, $x + h$, and the radii $r_h$ and $r_h + |h|$]
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Let \( r_h \) be a good radius for \( M_\beta f(x + h) = r_h^{\beta - d} \int_{B(x+h,r_h)} |f| \).

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\[ \leq (r_h + |h|)^{\beta - d - 1} \int_{B(x,r_h+|h|)} |f| \]
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\leq (r_h + |h|)^{\beta-d-1} \int_{B(x,r_h+|h|)} |f| \\
\leq M_{\beta-1} f(x)
\]
Finite differences

- If there is a finite constant $A$ such that
  
  \[ \|D_h f\|_{L^q} \leq A, \quad D_h f(x) = \frac{f(x + h) - f(x)}{|h|} \]

  for all $h \in \mathbb{R}^d$, then the weak derivatives of $f$ exist and

  \[ \|\nabla f\|_{L^q} \leq CA \]

  for a constant $C$ only depending on the dimension $d$ (for $1 < q < \infty$).
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- Kinnunen’s principle:
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- Kinnunen’s principle:
  - $S_\beta$ is sublinear: $S_\beta(f + g) \leq S_\beta f + S_\beta g$.
  - $S_\beta$ commutes with translations
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If there is a finite constant $A$ such that
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Kinnunen’s principle:
- $S_\beta$ is sublinear: $S_\beta(f + g) \leq S_\beta f + S_\beta g$.
- $S_\beta$ commutes with translations
\[ |D_h S_\beta f(x)| \leq |S_\beta D_h f(x)|. \]
So enough to establish
\[ \| S_\beta D_h f \|_q \lesssim \| f \|_p \]
uniformly in $h \in \mathbb{R}^d$. 
A Fourier transform look

\[ S_\beta D_h f(x) = \sup_{r > 0} |r^\beta \sigma_r * D_h f(x)| \]
A Fourier transform look

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\[ \widehat{D_h f}(\xi) = \frac{e^{i\xi \cdot h} - 1}{|h|} \hat{f}(\xi) = \frac{e^{i\xi \cdot h} - 1}{|h||\xi|} \hat{f}(\xi)|\xi| \]
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A Fourier transform look

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A Fourier transform look

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Then,

\[ S_\beta D_h f(x) = \sup_{r > 0} \left| \mathcal{F}^{-1} \left( r^{\beta - 1}(r|\xi|) \hat{\sigma}(r\xi) \mathcal{F}(T^h f) \right) (x) \right|. \]
A Fourier transform look

\[ S_\beta D_h f(x) = \sup_{r > 0} |r^\beta \sigma_r \ast D_h f(x)| \]

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Then,

\[ S_\beta D_h f(x) = \sup_{r > 0} \left| \mathcal{F}^{-1} \left( r^{\beta - 1} (r|\xi|) \tilde{\sigma}(r\xi) \mathcal{F}(T^h f) \right) (x) \right|. \]

Note

\[ |\partial^\gamma a^h(\xi)| \lesssim |\xi|^{-|\gamma|} \quad \text{for all multi-indexes } \gamma \in \mathbb{N}_0^d, \]
A Fourier transform look

\[ S_{\beta} D_h f(x) = \sup_{r > 0} |r^\beta \sigma_r \ast D_h f(x)| \]

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Then,

\[ S_{\beta} D_h f(x) = \sup_{r > 0} \left| \mathcal{F}^{-1} \left( r^{\beta - 1} (r|\xi|) \sigma_r(\xi) \mathcal{F}(T^h f) \right)(x) \right|. \]

Note

\[ |\partial^\gamma a^h(\xi)| \lesssim |\xi|^{-|\gamma|} \quad \text{for all multi-indexes } \gamma \in \mathbb{N}_0^d, \]

\( T_h \) is bounded on \( L^p \), uniformly in \( h \in \mathbb{R}^d \) for all \( 1 < p < \infty \) (Mikhlin–Hörmander multiplier).
The Fourier transform of the spherical measure is

\[ \hat{\sigma}(\xi) = 2\pi |\xi|^{-\frac{d-2}{2}} J_{\frac{d-2}{2}} (2\pi |\xi|) = \sum_{\pm} a_{\pm}(\xi) e^{\pm 2\pi i |\xi|}, \]

where

\[ |\partial_\xi^\gamma a_{\pm}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2} - |\gamma|} \]

for all multi-indices \( \gamma \in \mathbb{N}_0^d \).
Spherical measure

The Fourier transform of the spherical measure is

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When studying \( (r|\xi|)\hat{\sigma}(r\xi) \):

- Low/high frequencies depend on \( r \): \( |\xi| \sim 1/r \).
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When studying \((r|\xi|)\hat{\sigma}(r\xi)\):

- Low/high frequencies depend on \(r\): \(|\xi| \sim 1/r\).
- \((r|\xi|)\) helps with low frequencies \(\rightarrow\) decay.
- \((r|\xi|)\) gives one order Fourier growth for high freq \(\rightarrow\) \(d\) “replaced” by \(d - 2\).
Single scale reduction for \((q \geq 2)\)

Want to fix \(r\):

\[
\sup_{r > 0} \sup_{k \in \mathbb{Z}} \sup_{2^{-k} < r \leq 2^{-k+1}} \left( \sum_{k \in \mathbb{Z}} \left| \sup_{2^{-k} < r \leq 2^{-k+1}} |q| \right) \right)^{1/q} \leq \left( \sum_{k \in \mathbb{Z}} \left( \sum_{2^{-k} < r \leq 2^{-k+1}} |q| \right)^{1/q} \right)^{1/q}
\]
Single scale reduction for \( q \geq 2 \)

Want to fix \( r \):

\[
\sup_{r > 0} \sup_{k \in \mathbb{Z}} \sup_{2^{-k} \leq r \leq 2^{-(k+1)}} |q|^{1/q} \leq \left( \sum_{k \in \mathbb{Z}} \sup_{2^{-k} \leq r \leq 2^{-(k+1)}} |q| \right)^{1/q}
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Want to quantify \(|\xi| \sim 2^j\) to make effective use of decay/growth of multiplier.
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Want to quantify \(|\xi| \sim 2^j\) to make effective use of decay/growth of multiplier.

If

\[
\left\| \sup_{1 \leq r \leq 2} \left| \mathcal{F}^{-1}(r^{\beta - 1}(r|\xi|)\hat{\sigma}(r\xi)\hat{f}_j) \right| \right\|_{L^q} \lesssim \left( 2^{js_1}1\{j \leq 0\} + 2^{-js_2}1\{j > 0\} \right) \|f_j\|_{L^p}
\]

for \(s_1, s_2 > 0\),
Single scale reduction for \( (q \geq 2) \)

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\sup_{r>0} \sup_{k \in \mathbb{Z}} \sup_{2^{-k} \leq r \leq 2^{-k+1}} \leq \left( \sum_{k \in \mathbb{Z}} \sup_{2^{-k} \leq r \leq 2^{-k+1}} |q| \right)^{1/q}
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\]

for \( s_1, s_2 > 0 \), rescaling gives

\[
\left\| \sup_{2^{-k} \leq r \leq 2^{-k+1}} \left| \mathcal{F}^{-1} (r^{\beta-1} (r|\xi|) \hat{\sigma}(r\xi) \hat{f}_{j+k}) \right| \right\|_{L^q} \lesssim (2^{js_1} 1_{\{j \leq 0\}} + 2^{-js_2} 1_{\{j > 0\}}) \|f_{j+k}\|_{L^p}
\]

under the relation \( \frac{1}{q} = \frac{1}{p} - \frac{\beta-1}{d} \).
Single scale reduction for \((q \geq 2)\)

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under the relation \(\frac{1}{q} = \frac{1}{p} - \frac{\beta-1}{d}\).

Decay in \(j\) and Littlewood–Paley theory in \(k\) allows one to sum.
Lee’s interpolation scheme \((d \geq 5)\)

The \(L^p - L^q\) estimates for \(S^1f \coloneqq \sup_{1 \leq r \leq 2} |\sigma_r \ast f|\) when \(d \geq 3\) follow from interpolating

\[
\begin{align*}
\|S^1 f_j\|_{L^1} &\lesssim 2^j \|f_j\|_{L^1} \\
\|S^1 f_j\|_{L^\infty} &\lesssim 2^j \|f_j\|_{L^1} \\
\|S^1 f_j\|_{L^\infty} &\lesssim \|f_j\|_{L^\infty} \\
\|S^1 f_j\|_{L^2} &\lesssim 2^{-\frac{d-2}{2}j} \|f_j\|_{L^2} \\
\|S^1 f_j\|_{L^{\frac{2(d+1)}{d-1}}} &\lesssim 2^{-j \frac{d^2-2d-1}{2d+2}} \|f_j\|_{L^2}.
\end{align*}
\]

and getting the values for \(p\) and \(q\) such that

\[
\|S^1 f_j\|_{L^q} \lesssim 2^{-j \varepsilon(p,q)} \|f_j\|_{L^p}.
\]
Lee’s interpolation scheme \((d \geq 5)\)

The \(L^p - L^q\) estimates for \(S^{\sim 1} f := \sup_{1 \leq r \leq 2} |\sigma_r \ast f|\) when \(d \geq 3\) follow from interpolating

\[
\|S^{\sim 1} f_j\|_{L^1} \lesssim 2^j \|f_j\|_{L^1}
\]
\[
\|S^{\sim 1} f_j\|_{L^\infty} \lesssim 2^j \|f_j\|_{L^1}
\]
\[
\|S^{\sim 1} f_j\|_{L^\infty} \lesssim \|f_j\|_{L^\infty}
\]
\[
\|S^{\sim 1} f_j\|_{L^2} \lesssim 2^{-\frac{d-2}{2} j} \|f_j\|_{L^2}
\]
\[
\|S^{\sim 1} f_j\|_{L^{\frac{2(d+1)}{d-1}}} \lesssim 2^{-j \frac{d^2-2d-1}{2d+2}} \|f_j\|_{L^2}.
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Bounds for \(f \mapsto \sup_{1 \leq r \leq 2} |\mathcal{F}^{-1}((r|\xi|)\hat{\sigma}(r\xi)\hat{f})|\) follow from bounds for

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\]

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\|S^1 f_j\|_{L^\infty} \lesssim 2^j \|f_j\|_{L^1} \quad \|S^1 g_j\|_{L^\infty} \lesssim 2^{2j} \|f_j\|_{L^1}
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\]

\[
\|S^1 f_j\|_{L^2} \lesssim 2^{-\frac{d-2}{2}j} \|f_j\|_{L^2} \quad \|S^1 g_j\|_{L^2} \lesssim 2^{-\frac{d-4}{2}j} \|f_j\|_{L^2}
\]

\[
\|S^1 f_j\|_{L^\frac{2(d+1)}{d-1}} \lesssim 2^{-j \frac{d^2-2d-1}{2d+2}} \|f_j\|_{L^2} \quad \|S^1 g_j\|_{L^\frac{2(d+1)}{d-1}} \lesssim 2^{-j \frac{d^2-4d-3}{2d+2}} \|f_j\|_{L^2}
\]

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\|S^1 f_j\|_{L^q} \lesssim 2^{-j \epsilon(p,q)} \|f_j\|_{L^p}.
\]

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\[
f \mapsto S^1 g, \quad \text{where} \quad \hat{g}(\xi) = \hat{f}(\xi) |\xi|.
\]
Lee’s interpolation scheme \((d \geq 5)\)

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\|S^1 f\|_{L^{\frac{2(d+1)}{d-1}}} &\lesssim 2^{-j \frac{d^2-2d-1}{2d+2}} \|f\|_{L^2}.
\end{align*}
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Bounds for \(f \mapsto \sup_{1 \leq r \leq 2} |\mathcal{F}^{-1}((r|\xi|)\hat{\sigma}(r\xi)\hat{f})|\) follow from bounds for

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Lee’s interpolation scheme ($d \geq 5$)

The $L^p - L^q$ estimates for $S^{1}f := \sup_{1 \leq r \leq 2} |\sigma_{r} * f|$ when $d \geq 3$ follow from interpolating

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\|S^{1}f_{j}\|_{L^{1}} \lesssim 2^{j} \|f_{j}\|_{L^{1}} \quad \|S^{1}g_{j}\|_{L^{1}} \lesssim 2^{2j} \|f_{j}\|_{L^{1}}
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\]
\[
\|S^{1}f_{j}\|_{L^{\infty}} \lesssim \|f_{j}\|_{L^{\infty}}
\]
\[
\|S^{1}f_{j}\|_{L^{2}} \lesssim 2^{\frac{d-2}{2}j} \|f_{j}\|_{L^{2}} \quad \|S^{1}g_{j}\|_{L^{2}} \lesssim 2^{2j} \|f_{j}\|_{L^{2}}
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\[
\|S^{1}f_{j}\|_{L^{\frac{2(d+1)}{d-1}}} \lesssim 2^{-j \frac{d^{2}-2d-1}{2d+2}} \|f_{j}\|_{L^{2}}. \quad \|S^{1}g_{j}\|_{L^{\frac{2(d+1)}{d-1}}} \lesssim 2^{-j \frac{d^{2}-4d-3}{2d+2}} \|f_{j}\|_{L^{2}}.
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Bounds for $f \mapsto \sup_{1 \leq r \leq 2} |\mathcal{F}^{-1}((r|\xi|)\hat{\sigma}(r\xi)\hat{f})|$ follow from bounds for

$f \mapsto S^{1}g$, where $\hat{g}(\xi) = \hat{f}(\xi)|\xi|$.
Recall

$$\hat{\sigma}(\xi) = \sum_{\pm} a_{\pm}(\xi) e^{\pm 2\pi i |\xi|}, \quad \sigma_r * f(x) = \sum_{\pm} \int_{\mathbb{R}^d} e^{2\pi i (\xi \cdot x \pm |\xi| r)} a_{\pm}(r \xi) \hat{f}(\xi) \, d\xi.$$
The local smoothing phenomenon

Recall

\[ \hat{\sigma}(\xi) = \sum_{\pm} a_{\pm}(\xi) e^{\pm 2\pi i |\xi|}, \quad \sigma_r * f(x) = \sum_{\pm} \int_{\mathbb{R}^d} e^{2\pi i (\xi \cdot x \pm |\xi| r)} a_{\pm}(r\xi) \hat{f}(\xi) \, d\xi. \]

Half-wave propagator

\[ e^{it\sqrt{-\Delta}} f(x) := \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{it|\xi|} \hat{f}(\xi) \, d\xi. \]
The local smoothing phenomenon

Recall

\( \hat{\sigma}(\xi) = \sum_{\pm} a_{\pm}(\xi) e^{\pm 2\pi i |\xi|} \), \hspace{1cm} \sigma_r * f(x) = \sum_{\pm} \int_{\mathbb{R}^d} e^{2\pi i (\xi \cdot x \pm |\xi| r)} a_{\pm}(r\xi) \hat{f}(\xi) \, d\xi.

Half-wave propagator

\( e^{it\sqrt{-\Delta}} f(x) := \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{it|\xi|} \hat{f}(\xi) \, d\xi. \)

For any fixed time \( t \) and any \( 1 < p < \infty \), Peral (1980, also Miyachi) proved that

\[ \| e^{it\sqrt{-\Delta}} f \|_{L^p_{-s_p}(\mathbb{R}^d)} \leq C_{t,p} \| f \|_{L^p(\mathbb{R}^d)} \]

for \( s_p := (d - 1)|1/2 - 1/p| \) and \( C_{t,p} \) locally bounded in \( t \).
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State of the art for the local smoothing conjecture

- Mockenhoupt–Seeger–Sogge
- Wolff (d = 2)
- Wolff–Laba
- Garrigós–Seeger (–Schlag)
- Heo–Nazarov–Seeger (d ≥ 4)
- Bourgain–Demeter
- S. Lee–Vargas (d = 2)
- J. Lee (d = 2)

\[
\left( \int_1^2 \| e^{it\sqrt{-\Delta}} f \|_{L^p_{s - sp + \theta}}^p \, dt \right)^{1/p} \lesssim \| f \|_{L^p_s(\mathbb{R}^d)}
\]

holds for \( 0 \leq \theta < \frac{1}{p} \) and \( s_p = (d - 1)\left(\frac{1}{2} - \frac{1}{p}\right) \) whenever \( p \geq \frac{2(d+1)}{d-1} \).

\[
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For frequency localised pieces it reads as

\[
\left( \int_1^2 \| e^{it\sqrt{-\Delta}} f_j \|_{L^p(\mathbb{R}^d)}^p \, dt \right)^{1/p} \lesssim 2^{j(s_p-\theta)} \| f \|_{L^p(\mathbb{R}^d)}.
\]

\[
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Using this at \(p = d-1\) with the Sobolev embedding \(\left\| u \right\|_{L^\infty_t(\mathbb{R})} \lesssim \left\| (1+\sqrt{-\partial^2_t})^{s} u \right\|_{L^p_t(\mathbb{R})}\) for \(s > 1/p\), one has

\[
\left\| S^{\sim 1} g_j \right\|_{L^{d-1}} = \left\| \sup_{1 \leq t \leq 2} |\sigma_t \ast g_j| \right\|_{L^{d-1}} \lesssim \delta \ 2^{j(\delta-1)} \left\| g_j \right\|_{L^{d-1}} \lesssim 2^{j \delta} \left\| f_j \right\|_{L^{d-1}}
\]

for any \(\delta > 0\).
Endpoint Sobolev bounds for HL and $0 < \beta < 1$

For $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$, $1 < p \leq q < \infty$ and $0 \leq \beta < d$

$$\| \nabla M_\beta f \|_{L^q(\mathbb{R}^d)} \leq \| M_\beta |\nabla f| \|_{L^q(\mathbb{R}^d)} \leq \| \nabla f \|_{L^p(\mathbb{R}^d)}.$$
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$$\| \nabla M_\beta f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \leq \| M_{\beta-1} f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \leq \| f \|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \| \nabla f \|_{L^1(\mathbb{R}^d)}.$$
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For $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$, $1 < p \leq q < \infty$ and $0 \leq \beta < d$

$$\| \nabla M_{\beta} f \|_{L^q(\mathbb{R}^d)} \leq \| M_{\beta} |\nabla f| \|_{L^q(\mathbb{R}^d)} \leq \| \nabla f \|_{L^p(\mathbb{R}^d)}. $$

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- B.–Ramos–Saari (2018), for any $d > 1$, centered $M_\beta$, but supremum restricted to $r = 2^k$ and $k \in \mathbb{Z}$, or for maximal functions with smooth kernels,

$$M_{\beta}^\phi f(x) = \sup_{r>0} r^\beta |f| \ast r^{-d} \phi_{B(0,r)}(x).$$
Endpoint Sobolev bounds for “HL” and $0 < \beta < 1$

Inspired by the $\beta \geq 1$ case:

$$\|\nabla M_\beta f\|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \leq \|M_{\beta-1} f\|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \leq \|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$
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Using similar Fourier analytic arguments as before, one may observe

$$\| \nabla M_\beta^\sim f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \lesssim \| f \|_{\dot{B}^{1-\beta}_{d,d-\beta,1}(\mathbb{R}^d)} , \quad 0 < \beta < d/2 ,$$

where

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This, “nearly” upgrades to the full supremum as

$$\left\| \nabla M_{\beta} f \right\|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \lesssim \left\| f \right\|_{\dot{B}^{1-\beta \frac{d}{d-\beta}, \frac{d}{d-\beta}}_{d-\beta, d-\beta}(\mathbb{R}^d)}.$$  

(1)
Inspired by the $\beta \geq 1$ case:
\[
\| \nabla M_\beta f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \leq \| M_{\beta-1} f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \leq \| f \|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \lesssim \| \nabla f \|_{L^1(\mathbb{R}^d)}.
\]
Using similar Fourier analytic arguments as before, one may observe
\[
\| \nabla \tilde{M}_\beta^{-1} f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \lesssim \| f \|_{\dot{B}^{1-\beta}_{\frac{d}{d-\beta},1}(\mathbb{R}^d)}, \quad 0 < \beta < d/2,
\]
where
\[
\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| f_j \|_{L^p(\mathbb{R}^d)}^q \right)^{1/q}.
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This, “nearly” upgrades to the full supremum as
\[
\| \nabla M_\beta f \|_{L^{\frac{d}{d-\beta}}(\mathbb{R}^d)} \lesssim \| f \|_{\dot{B}^{1-\beta}_{\frac{d}{d-\beta}+\frac{d}{d-\beta}}(\mathbb{R}^d)}. \tag{1}
\]
Compare (1) with the pointwise Kinnunen–Saksman estimate
\[
|\nabla M_\beta f(x)| \lesssim M_{\beta-1} f(x).
\]
Inspired by the $\beta \geq 1$ case:

$$
\| \nabla M_\beta f \|_{L^{d/(d-\beta)}(\mathbb{R}^d)} \leq \| M_{\beta-1} f \|_{L^{d/(d-\beta)}(\mathbb{R}^d)} \leq \| f \|_{L^{d/(d-1)}(\mathbb{R}^d)} \lesssim \| \nabla f \|_{L^1(\mathbb{R}^d)}.
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Using similar Fourier analytic arguments as before, one may observe

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- $M$ is sublinear: $M(f + g) \leq M(f) + M(g)$. Boundedness implies continuity.
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The map

$$f \mapsto |\nabla Mf|$$

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Happy Birthday Andreas!