Hausdorff dimension of Furstenberg-type sets associated to families of affine subspaces

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Unions of affine subspaces

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Definition

By the union of an $s$-dimensional collection of affine subspaces in $\mathbb{R}^n$ we mean a union $\bigcup_{P \in E} P \subset \mathbb{R}^n$, where $\emptyset \neq E \subset A(n, k)$ with $\dim E = s$.

Here and in the sequel dimension always refers to Hausdorff dimension.

We will study the following question:

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How large must the union $\bigcup_{P \in E} P$ of affine subspaces be depending on the dimension of $E$?
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Large families of affine subspaces

**Theorem (Oberlin (2014))**

The union of any nonempty \(s\)-dimensional compact family of \(k\)-dimensional affine subspaces in \(\mathbb{R}^n\) has

- dimension at least \(s + 2k - k(n - k)\) if \(s \in [0, (k + 1)(n - k) - k]\),
- positive Lebesgue-measure if \(s \in ((k + 1)(n - k) - k, (k + 1)(n - k)]\).
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The dimension bound is sharp for $s \in [(k + 1)(n - k) - k - 1, (k + 1)(n - k) - k]$. 

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In the case of hyperplanes, Oberlin’s theorem states the following:
Families of affine subspaces

Theorem (Oberlin (2007))

The union of any nonempty $s$-dimensional compact family of hyperplanes in $\mathbb{R}^n$ has

- dimension exactly $s + n - 1$ if $s \in [0, 1]$,
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Lemma

For any integers $1 \leq k < n$ and any $s \geq 0$, if $E$ is a nonempty $s$-dimensional family of $k$-dimensional affine subspaces then $\dim \left( \bigcup_{P \in E} P \right) \leq k + s$.

Proof: $E$ can be decomposed into finitely many parts $E_i$ such that for each $i$, $\bigcup_{P \in E_i} P$ is obtained as a Lipschitz image of $E_i \times \mathbb{R}^k$, and then

$$\dim \left( \bigcup_{P \in E} P \right) = \max_i \dim \left( \bigcup_{P \in E_i} P \right) \leq \max_i (\dim E_i + k) = s + k.$$
Small families of affine subspaces

We obtained the following result for $k$-dimensional affine subspaces in the $s \in [0, 1]$ case:

**Theorem (H., Keleti, Máthé, 2017)**

For any $s \in [0, 1]$, the union of any nonempty $s$-dimensional family of $k$-dimensional affine subspaces of $\mathbb{R}^n$ has dimension exactly $s + k$. 

On the other hand, the condition $s \in [0, 1]$ in our theorem can not be dropped: For any collection of $k$-planes of a fixed $(k+1)$-plane of $\mathbb{R}^n$ the union clearly has Hausdorff dimension at most $k+1$, which is less than $s + k$ if $s > 1$.

For general families of affine subspaces, we proved the following:

**Theorem (H., Keleti, Máthé, 2017)**

For any $s \in [0, \frac{(k+1)(n-k)}{2}]$, the union of any nonempty $s$-dimensional family of $k$-dimensional affine subspaces of $\mathbb{R}^n$ has dimension at least $k + \min(s, 1)$. This is sharp for $s \in [0, k+1]$. 

In the proof we use an integration technique similar to Cordoba's proof for the Kakeya maximal conjecture in the plane.
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General families of affine subspaces

Using a different method based on combinatorial-geometric arguments, the following was proved:

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For any \( s \in \left[0, (k + 1)(n - k)\right] \), the union of any nonempty \( s \)-dimensional family of \( k \)-dimensional affine subspaces of \( \mathbb{R}^n \) has dimension at least \( k + \frac{s}{k+1} \).
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This is sharp for $s = m(k + 1)$, where $m \in [0, n - k]$ is any integer, and almost sharp in general, namely, there exist suitable unions of affine subspaces such that the gap between their dimension and the dimension bound obtained above is less than 1.
Dimension bounds and sharpness in the $k = 2, n = 8$ case

\[ f(s) = \max\{\min\{s - 8, 8\}, 2 + \min\{s, 1\}\}, \text{ gray}, \]
\[ g(s) = 2 + \frac{s}{3}, \text{ blue}, \]
\[ h(s) = \begin{cases} 
  s - 2\lceil\frac{s}{3}\rceil + 4 & \text{if } \lceil\frac{s}{3}\rceil \geq \frac{2+s}{3}, \\
  2 + \lceil\frac{s}{3}\rceil & \text{if } \lceil\frac{s}{3}\rceil \leq \frac{2+s}{3}
\end{cases}, \text{ red dashed}. \]
Collections of large subsets of affine subspaces

What can we say if instead of full subspaces we only have a large subset of each subspace?

Theorem (Falconer and Mattila (2016))

If $E$ is a nonempty $s$-dimensional Borel family of hyperplanes and $B \subset \mathbb{R}^n$ is Borel such that

$$\lambda_n^r(B \cap P) > 0 \quad \text{for all} \quad P \in E,$$

then

$$\dim(\bigcup_{P \in E} P \cap B) = \dim(\bigcup_{P \in E} P) = s + n - 1$$

if $s \in [0,1]$.

We can go further: it is enough to take a $k$-dimensional subset of each of the affine subspaces:

Theorem (H., Keleti, Máté, 2017)

Let $1 \leq k < n$ be integers and $s \in [0,1]$. If $E$ is a nonempty $s$-dimensional family of $k$-dimensional affine subspaces and $B \subset \mathbb{R}^n$ such that $\dim(B \cap P) = k$ for every $P \in E$ then

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If $E$ is a nonempty $s$-dimensional Borel family of hyperplanes and $B \subset \mathbb{R}^n$ is Borel such that $\mathcal{L}^{n-1}(B \cap P) > 0$ for all $P \in E$, then

- $\dim (\bigcup_{P \in E} P) \cap B = \dim (\bigcup_{P \in E} P) = s + n - 1$ if $s \in [0, 1]$
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Applications for skeletons of cubes

Let $0 \leq k < n$ be integers, and $C \subset \mathbb{R}^n$ an $n$-dimensional cube. By the $k$-skeleton of the cube $C$ we mean the union of the $k$-faces of $C$. 

Question

Suppose that $B \subset \mathbb{R}^n$ contains the $k$-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$. What can we say about the size of $B$?

Corollary (H., Keleti, Máté, 2017)

For any integers $0 \leq k < n$, if $B \subset \mathbb{R}^n$ contains the $k$-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$, then $\dim B \geq k + 1$.

Idea of proof:

Define $E \subset A(n,k)$ as the set of the $k$-dimensional affine subspaces of the $k$-faces of the cubes contained in $B$. It can be shown that the condition that $B \subset \mathbb{R}^n$ contains the $k$-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$ implies that we have $\dim E \geq 1$. Thus by our previous theorem, $\dim B \geq k + \min(\dim E, 1) = k + 1$.

Remark: In the $k = 0$ case the proof is simpler.
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Theorem (Chang, Csörnyei, H., Keleti, 2017)

For any integers $0 \leq k < n$, there exists a Borel set $B \subset \mathbb{R}^n$ with Hausdorff dimension $k + 1$ that contains the $k$-skeleton of a rotated unit cube centered at every point of $\mathbb{R}^n$.

In fact, we show that in an appropriate Baire category sense, a typical construction has dimension $k + 1$.

Other interesting results about cube skeletons can be found in the papers:

- Tamás Keleti, Daniel Nagy, Pablo Shmerkin: Squares and their centers
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Question: What can we say about \( \dim B \) if instead of \( \dim (B \cap P) = k \) we have \( \dim (B \cap P) \geq \alpha \) for all \( P \in E \) (\( 0 < \alpha \leq k \)??)
Furstenberg-type sets

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Such sets can be seen as generalized Furstenberg-type sets.
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**Classical Furstenberg sets (origin: Furstenberg, 1969)**

Let $0 < \alpha \leq 1$. A Borel set $F \subset \mathbb{R}^2$ is an $\alpha$-Furstenberg set, if for every $e \in S^1$ there is a line $L_e$ in direction $e$ such that $\dim (L_e \cap F) \geq \alpha$. 
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- If $F$ is an $\alpha$-Furstenberg set, then $\dim F \geq 2\alpha$, $\dim F \geq \alpha + \frac{1}{2}$ (Wolff, 2003).
- There exists an $\alpha$-Furstenberg set with $\dim F = \frac{3\alpha}{2} + \frac{1}{2}$ (Wolff, 2003).
- If $F$ is a $1/2$-Furstenberg set, then $\dim F \geq 1 + c$ for an absolute constant $c > 0$ (Bourgain, 2003).
Furstenberg-type sets

Generalized Furstenberg sets

Let $0 < \alpha \leq 1$, $0 < s \leq 1$. A Borel set $F \subset \mathbb{R}^2$ is an $(\alpha, s)$-Furstenberg set, if there exists $\emptyset \neq E \subset S^1$ with $\dim E = s$ such that for every $e \in E$ there is a line $L_e$ in direction $e$ such that $\dim (L_e \cap F) \geq \alpha$. 

Theorem (Molter and Rela, 2012)

Let $0 < \alpha \leq 1$, $0 < s \leq 1$, and let $F \subset \mathbb{R}^2$ be an $(\alpha, s)$-Furstenberg set. Then $\dim F \geq 2 \alpha - 1 + s$, and $\dim F \geq \alpha + s^2$. The proof is a generalization of Wolff’s proof for classical Furstenberg sets.

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Let $0 < \alpha \leq 1$, $0 < s \leq 1$, and let $F \subset \mathbb{R}^2$ be an $(\alpha, s)$-Furstenberg set. Then $\dim F \geq \alpha + \min(s, \alpha)$. This is sharp for each $\alpha, s$ with $s \leq \alpha$. 

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Furstenberg-sets associated to families of affine subspaces

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Let $0 < k < n$ be integers, $0 < \alpha \leq k$, $0 \leq s \leq (k + 1)(n - k)$. We say that $B \subset \mathbb{R}^n$ is an $(\alpha, k, s)$-Furstenberg set, if there exists $\emptyset \neq E \subset A(n, k)$ with $\dim E = s$ such that for every $P \in E$, $\dim (P \cap B) \geq \alpha$.
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**Theorem A (H., Keleti, Máthé, 2017)**

Let $1 \leq k < n$ be integers, and $0 < \alpha \leq k$, $0 \leq s \leq (k + 1)(n - k)$ be any real numbers. If $B \subset \mathbb{R}^n$ is an $(\alpha, k, s)$-Furstenberg set, then $\dim B \geq 2\alpha - k + \min(s, 1)$. Clearly, Theorem A implies one of our previous results for unions of affine subspaces. Theorem A is a generalization of the first estimate of Molter and Rela, and thus also a generalization of the first Furstenberg estimate: If $F \subset \mathbb{R}^2$ is a classical $\alpha$-Furstenberg set, then the lines $L_e$ form an at least 1-dimensional collection of lines of $\mathbb{R}^2$ and thus $\dim F \geq 2\alpha - 1 + 1 = 2\alpha$. 

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Theorem B (H., 2018)

Let $1 \leq k < n$ be integers, and $k - 1 < \alpha \leq k$, $0 \leq s \leq (k + 1)(n - k)$ be any real numbers. If $B \subset \mathbb{R}^n$ is an $(\alpha, k, s)$-Furstenberg set, then $\dim B \geq \alpha + \frac{s}{k + 1}$.
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If $\alpha \leq k - 1$, then this can not hold:

let $B$ be an $\alpha$-dimensional subset of a fixed $(k - 1)$-dimensional affine subspace $V$ and take a 1-dimensional family of $k$-dimensional affine subspaces containing $V$.

Then $\dim (P \cap B) = \alpha$ for all $P$, and $\dim B = \alpha < \alpha + \frac{1}{k+1}$. 
### Furstenberg-sets associated to families of affine subspaces

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The previous example shows that this is sharp.
The outline of the proofs of Theorems A and B

Theorem A (H., Keleti, Máthé, 2017)

Let $1 \leq k < n$ be integers, and $0 < \alpha \leq k$, $0 \leq s \leq (k + 1)(n - k)$ be any real numbers. If $B \subset \mathbb{R}^n$ is an $(\alpha, k, s)$-Furstenberg set, then $\dim B \geq 2\alpha - k + \min(s, 1)$.

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Remark: we don’t need any measurability condition on $E$ or $B$. Using a technique from descriptive set theory, it can be shown that the problem for general $E$ and $B$ can be reduced to the same problem for Borel $E$ and $B$. 
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#### Theorem A (H., Keleti, Máthé, 2017)

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- Both proofs can be considered as extensions of Wolff’s proof for classical Furstenberg-sets.
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- Remark: we don’t need any measurability condition on $E$ or $B$. Using a technique from descriptive set theory, it can be shown that the problem for general $E$ and $B$ can be reduced to the same problem for Borel $E$ and $B$.
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- Here we present the outline of the proof of both theorems for the Minkowski dimension of $B$. In the proofs we use a pigeonholing argument which makes it possible to estimate the Hausdorff dimension of $B$ instead of the Minkowski dimension.
The outline of the proof of Theorem A

The proof is based on an $L^2$-estimate, similarly to Cordoba’s proof for the Kakeya maximal conjecture in the plane.
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We need some geometric considerations. Fix a bounded set $S$ (e.g. a cube), and cut each affine subspace $P$ with $S$. Then the $\delta$-neighborhood of $P \cap S$ is called a $\delta$-tube. The Lebesgue measure of a $\delta$-tube is $\approx \delta^{n-k}$.
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- If $P$ and $P'$ are transversal, then $\mathcal{L}^n(P_\delta \cap P'_\delta \cap S) \approx \delta^{n-k+1}$.
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- If $P$ and $P'$ are transversal, then $\mathcal{L}^n(P_\delta \cap P'_\delta \cap S) \approx \delta^{n-k+1}$.
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- In general, $\mathcal{L}^n(P_\delta \cap P'_\delta \cap S) \lesssim \frac{\delta^{n-k+1}}{d(P,P')+\delta}$, where $d$ denotes the distance on $E \subset A(n, k)$. 
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- In general, $\mathcal{L}^n(P_\delta \cap P'_\delta \cap S) \lesssim \frac{\delta^{n-k+1}}{d(P,P')+\delta}$, where $d$ denotes the distance on $E \subset A(n,k)$.
- Most of the pairs of $\delta$-tubes intersect in a small enough set. Those pairs which have big intersection are rare enough, namely, for any fixed $P'$, the amount of affine subspaces $P$ which are nearly parallel to $P'$ (and intersect it) is small enough.
The outline of the proof of Theorem A

- It can be assumed without loss of generality that there is an $\varepsilon > 0$ such that $\mathcal{H}_\infty^\alpha(P \cap B) \geq \varepsilon$ for all $P \in E$, where $\mathcal{H}_\infty^\alpha$ denotes the $\alpha$-dimensional Hausdorff content. This implies $\mathcal{L}^n((P \cap B)_\delta) \gtrsim \delta^{n-\alpha}$. 

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- We fix a Frostman measure \( \mu \) on \( E \subset A(n, k) \). That is, there exists a constant \( C \) such that \( \mu(B(P, r)) \leq Cr^s \) for each \( P \in E, r > 0 \).
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- Consider the integral \( \int_E \mathcal{L}^n((P \cap B)_\delta) d\mu(P) \).
The outline of the proof of Theorem A

- It can be assumed without loss of generality that there is an $\varepsilon > 0$ such that $\mathcal{H}_\alpha^\infty(P \cap B) \geq \varepsilon$ for all $P \in E$, where $\mathcal{H}_\alpha^\infty$ denotes the $\alpha$-dimensional Hausdorff content. This implies $\mathcal{L}^n((P \cap B)_\delta) \gtrsim \delta^{n-\alpha}$.

- We fix a Frostman measure $\mu$ on $E \subset A(n, k)$. That is, there exists a constant $C$ such that $\mu(B(P, r)) \leq Cr^s$ for each $P \in E$, $r > 0$.

- Consider the integral $\int_E \mathcal{L}^n((P \cap B)_\delta) d\mu(P)$.
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  - We have that $\int_E \mathcal{L}^n((P \cap B)_\delta) d\mu(P) \gtrsim \delta^{n-\alpha}$.
  - To get an upper bound, we apply the Cauchy-Schwarz inequality for the functions $y \mapsto \chi_{B_\delta}(y)$, $y \mapsto \int_E \chi_{P_\delta}(y) d\mu(P)$.
The outline of the proof of Theorem A

We get

\[
\int_{E} \mathcal{L}^n((P \cap B_\delta)) d\mu(P) \leq \int_{\mathbb{R}^n} \chi_{B_\delta}(y) \cdot \int_{E} \chi_{P_\delta}(y) d\mu(P) dy \leq
\]

\[
\left(\int_{\mathbb{R}^n} \chi_{B_\delta}^2(y) dy \right)^{1/2} \cdot \left(\int_{\mathbb{R}^n} \left(\int_{E} \chi_{P_\delta}(y) d\mu(P)\right)^2 dy\right)^{1/2} =
\]

\[
= (\mathcal{L}^n(B_\delta))^{1/2} \cdot \left(\int_{E \times E} \mathcal{L}^n(P_\delta \cap P'_\delta) d\mu(P) d\mu(P')\right)^{1/2}.
\]
The outline of the proof of Theorem A

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\[
\int_E \mathcal{L}^n((P \cap B)\delta) d\mu(P) \leq \int \chi_{B_\delta}(y) \cdot \int E \chi_{P_\delta}(y) d\mu(P) dy \leq
\]

\[
\left( \int \chi_{B_\delta}(y) dy \right) \cdot \left( \int \int E \chi_{P_\delta}(y) d\mu(P) dy \right) \]

\[
= (\mathcal{L}^n(B_\delta))^{1/2} \cdot \left( \int \int E \times E \mathcal{L}^n(P_\delta \cap P'_\delta) d\mu(P) d\mu(P') \right)^{1/2}.
\]

We need an upper bound for \( \int \int E \times E \mathcal{L}^n(P_\delta \cap P'_\delta) d\mu(P) d\mu(P') \).
The outline of the proof of Theorem A

It can be proved that for any fixed $P' \in E$,

$$
\int_E \mathcal{L}^n (P_\delta \cap P'_\delta) \, d\mu(P) \lesssim \delta^{n-k+s} \log \left( \frac{1}{\delta} \right)
$$

and thus

$$
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where $s = \min(\dim E, 1)$.
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It can be proved that for any fixed $P' \in E$,

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$$

where $s = \min(\dim E, 1)$. We get

$$
\mathcal{L}^n(B_{\delta}) \gtrsim \frac{\delta^{n-(2\alpha-k+s)}}{\log \left( \frac{1}{\delta} \right)}
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and this immediately implies $\dim_{B} B \geq 2\alpha - k + \min(\dim E, 1)$. 

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and this immediately implies $\dim_B B \geq 2\alpha - k + \min(\dim E, 1)$.

To estimate the Hausdorff dimension of $B$, we need to find a big enough subset $F \subset B$ which is covered by balls of approximately the same radii and estimate $\mathcal{L}^n(F_{\delta})$ from below.
The outline of the proof of Theorem B

**Theorem B (H., 2018)**

Let \( 1 \leq k < n \) be integers, and \( k - 1 < \alpha \leq k \), \( 0 \leq s \leq (k + 1)(n - k) \) be any real numbers. If \( B \subset \mathbb{R}^n \) is an \((\alpha, k, s)\)-Furstenberg set, then \( \dim B \geq \alpha + \frac{s}{k+1} \).
The outline of the proof of Theorem B

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Main ingredients of the proof for the Minkowski dimension of $B$:
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Main ingredients of the proof for the Minkowski dimension of \(B\):

- Consider the dyadic cubes of side length \(\delta = 2^{-\ell}\) intersecting \(B\). We need to give a lower bound on the number \(N_\ell\) of these cubes.
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Main ingredients of the proof for the Minkowski dimension of \(B\):

- Consider the dyadic cubes of side length \(\delta = 2^{-\ell}\) intersecting \(B\). We need to give a lower bound on the number \(N_\ell\) of these cubes.

- Choose a maximal \(\delta = 2^{-\ell}\)-separated subset \(E'\) of \(E \subset A(n, k)\). It can be deduced using a Frostman measure on \(E\) that \(|E'| \gtrsim 2^{\ell s}\), where \(s = \dim E\).
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Let $1 \leq k < n$ be integers, and $k - 1 < \alpha \leq k$, $0 \leq s \leq (k + 1)(n - k)$ be any real numbers. If $B \subset \mathbb{R}^n$ is an $(\alpha, k, s)$-Furstenberg set, then $\dim B \geq \alpha + \frac{s}{k+1}$.

Main ingredients of the proof for the Minkowski dimension of $B$:

- Consider the dyadic cubes of side length $\delta = 2^{-\ell}$ intersecting $B$. We need to give a lower bound on the number $N_\ell$ of these cubes.

- Choose a maximal $\delta = 2^{-\ell}$-separated subset $E'$ of $E \subset A(n, k)$. It can be deduced using a Frostman measure on $E$ that $|E'| \gtrsim 2^{\ell s}$, where $s = \dim E$.

- Using this discretization, we apply a suitable double-counting method to bound $N_\ell$ from below.
The outline of the proof of Theorem B

We need the following facts:

Fix a collection of \((k+1)\)-many dyadic cubes:

\[ D = \{ D_1, \ldots, D_{k+1} \} \]

such that the simplex \( S_D \) generated by the centers of the cubes has big enough H\(_k\)-measure:

\[ H_k(S_D) \gtrsim \ell - m_1 \]

for some positive number \( m_1 \).

Then the number of \( 2^{\ell - m} \)-separated affine subspaces in \( E' \) intersecting all the cubes \( D_1, \ldots, D_{k+1} \) (of side length \( 2^{\ell - m} \)) is \( \lesssim \ell m_2 \) (where \( m_2 \) is a positive function of \( m_1 \)).

On the other hand, it can be proved that for each affine subspace \( P \in E' \), the number of collections of \((k+1)\)-many dyadic cubes intersecting \( P \cap B \) such that \( H_k(S_D) \gtrsim \ell - m_1 \) is big enough, namely, at least \( 2^{\ell} \cdot \alpha \cdot (k+1) \cdot \ell - m_3 \) (where \( m_3 \) is a positive function of \( m_1 \)).

We define a finite set \( T \) consisting of pairs of the form \((P, D)\), where \( P \) is an affine subspace from \( E' \) and \( D \) is a collection of \((k+1)\)-many dyadic cubes with the above additional properties. Then we count the cardinality of \( T \) in two ways using the above facts.
The outline of the proof of Theorem B

We need the following facts:

- Fix a collection of \((k + 1)\)-many dyadic cubes: \(D = \{D_1, \ldots, D_{k+1}\}\) such that the simplex \(S_D\) generated by the centers of the cubes has big enough \(\mathcal{H}^k\)-measure: \(\mathcal{H}^k(S_D) \gtrsim \ell^{-m_1}\) for some positive number \(m_1\).
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- Then the number of \(2^{-\ell}\)-separated affine subspaces in \(E'\) intersecting all the cubes \(D_1, \ldots, D_{k+1}\) (of side length \(2^{-\ell}\)) is \(\lesssim \ell^{m_2}\) (where \(m_2\) is a positive function of \(m_1\)).
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We need the following facts:

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2. Then the number of \(2^{-\ell}\)-separated affine subspaces in \(E'\) intersecting all the cubes \(D_1, \ldots, D_{k+1}\) (of side length \(2^{-\ell}\)) is \(\lesssim \ell^{m_2}\) (where \(m_2\) is a positive function of \(m_1\)).

3. On the other hand, it can be proved that for each affine subspace \(P \in E'\), the number of collections of \((k + 1)\)-many dyadic cubes intersecting \(P \cap B\) such that \(H^k(S_D) \gtrsim \ell^{-m_1}\) is big enough, namely, at least \(2^{\ell \cdot \alpha \cdot (k+1)} \cdot \ell^{-m_3}\) (where \(m_3\) is a positive function of \(m_1\)).
The outline of the proof of Theorem B

We need the following facts:

- Fix a collection of $(k+1)$-many dyadic cubes: $D = \{D_1, \ldots, D_{k+1}\}$ such that the simplex $S_D$ generated by the centers of the cubes has big enough $\mathcal{H}^k$-measure: $\mathcal{H}^k(S_D) \gtrsim \ell^{-m_1}$ for some positive number $m_1$.

- Then the number of $2^{-\ell}$-separated affine subspaces in $E'$ intersecting all the cubes $D_1, \ldots, D_{k+1}$ (of side length $2^{-\ell}$) is $\lesssim \ell^{m_2}$ (where $m_2$ is a positive function of $m_1$).

On the other hand, it can be proved that for each affine subspace $P \in E'$, the number of collections of $(k+1)$-many dyadic cubes intersecting $P \cap B$ such that $\mathcal{H}^k(S_D) \gtrsim \ell^{-m_1}$ is big enough, namely, at least $2^{\ell \cdot \alpha \cdot (k+1) \cdot \ell^{-m_3}}$ (where $m_3$ is a positive function of $m_1$).

- We define a finite set $T$ consisting of pairs of the form $(P, D)$, where $P$ is an affine subspace from $E'$ and $D$ is a collection of $(k+1)$-many dyadic cubes with the above additional properties. Then we count the cardinality of $T$ in two ways using the above facts.
The outline of the proof of Theorem B

We obtain the following bounds for the cardinality of $T$:

$$|T| \lesssim N_\ell \cdot \ell^{m_2},$$

where $N_\ell$ is the number of dyadic $2^\ell$-cubes intersecting $B$.

$$|T| \gtrsim |E'| \cdot 2^{\ell} \cdot \alpha \cdot (k+1) \cdot \ell^{-m_3} \gtrsim 2^{\ell} \cdot \alpha \cdot (k+1) \cdot \ell^{-m_3}.$$ 

This implies that $|N_\ell| \gtrsim 2^{\ell} \cdot \alpha \cdot (k+1) \cdot \ell^{-m_4}$ with a positive number $m_4$, and the bound $\dim B \geq \alpha + s_{k+1}$ concludes.

To estimate the Hausdorff dimension of $B$, we need a slightly more elaborate argument.
The outline of the proof of Theorem B

We obtain the following bounds for the cardinality of $T$:

- $|T| \lesssim N_\ell^{k+1} \cdot \ell^{m_2}$, where $N_\ell$ is the number of dyadic $2^\ell$-cubes intersecting $B$.

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This implies that $N_{\ell} \gtrsim 2^\ell \cdot \alpha (k+1) \cdot \ell m_3$ with a positive number $m_4$, and the bound $\dim B \geq \alpha (k+1)$ concludes.
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This implies that $|N_{\ell}| \gtrsim 2^{\ell \cdot (\alpha + \frac{s}{k+1})} \cdot \ell^{-m_4}$ with a positive number $m_4$, 
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We obtain the following bounds for the cardinality of $T$:

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and the bound $\dim_B B \geq \alpha + \frac{s}{k+1}$ concludes.

To estimate the Hausdorff dimension of $B$, we need a slightly more elaborate argument.
Thank you for your attention.