Improved bounds for Kakeya in intermediate dimensions using semialgebraic geometry

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joint work with
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Madison Lectures in Fourier Analysis
Definition

A **Kakeya set** is a compact subset of $\mathbb{R}^n$ that contains a unit line segment in every direction.

Definition

A $\delta$-**tube** is the $\delta$-neighbourhood of a unit line segment.

Definition

A family $\mathcal{T}$ of $\delta$-tubes is **direction-separated** if the angle between the core lines of any pair is at least $\delta$.

Fact

If $\mathcal{T}$ is direction-separated, then $\# \mathcal{T} \leq C_n \delta^{-(n-1)}$. 
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Fact

If $\mathbb{T}$ is direction-separated, then $\#\mathbb{T} \leq \delta^{-(n-1)}$. 

Direction-separated tubes can be compressed by translation.

Besicovitch (1917) constructed a Kakeya set of zero Lebesgue measure.
But hopefully they cannot be compressed more than that

\( \mathbb{T}_E \) will denote any family of direction-separated \( \delta \)-tubes which are contained in \( E \subset B(0, 2) \subset \mathbb{R}^n \).

**Conjecture (Kakeya)**

*For all \( \varepsilon > 0 \), there is a constant \( C \) such that*

\[
\delta^{n-1} \# \mathbb{T}_E \leq C \delta^{-\varepsilon} |E|.
\]

\( (K_\varepsilon) \)

Think of \( E \) as the \( \delta \)-neighbourhood of the set of the previous slide.

\( (K_\varepsilon) \) implies that Kakeya sets have Minkowski dimension \( n - \varepsilon \).

So even though Kakeya sets can have zero measure, it is conjectured that they can be no smaller than that.

This was confirmed by Davies (1971) in the two-dimensional case.

For higher dimensions, we start by simplifying the problem....
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Conjecture (Kakeya)

*For all $\varepsilon > 0$, there is a constant $C$ such that*

$$\delta^{n-1} \# \mathcal{T}_E \leq C \delta^{-\varepsilon} |E|.$$  \hfill (K$\varepsilon$)

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Definition
The **semialgebraic sets** are finite unions of sets of the form
\[
\left\{ x \in \mathbb{R}^n : P_1(x), \ldots, P_k(x) \leq 0, \quad P_{k+1}(x), \ldots, P_m(x) = 0 \right\}
\]
where the \( P_j : \mathbb{R}^n \to \mathbb{R} \) are polynomials.

Definition
The **complexity** \( D \) is the sum of the degrees of all the polynomials.

Conjecture (Kakeya for semialgebraic sets)
For all \( D \in \mathbb{N} \), there is a constant \( C \) such that
\[
\delta^{n-1} \# T_E \leq C |E|
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whenever \( E \subset \mathbb{R}^n \) is a semialgebraic set of complexity \( \leq D \).
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Conjecture (Guth-Zahl)
For all \( D \in \mathbb{N} \), there is a constant \( C \) such that

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Theorem (Katz-R.)
*For all \(D \in \mathbb{N}\), there is a constant \(C\) such that*
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Theorem (PWA)

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*whenever \( E \subset \mathbb{R}^n \) is a semialgebraic set of complexity \( \leq D \).*
Proof that $\delta^{n-1} \# \mathcal{T}_E \leq C|E|$ whenever $E$ is semialgebraic

Instead of discrete $\mathcal{T}_E$, we consider the continuum of lines in $E$;

$$L := \{ (a, d) \in \mathbb{R}^{2(n-1)} : (a + td, t) \in E \quad \forall \ t \in [1/2, 1] \}.$$

Tarski’s theorem implies that $L$ is semialgebraic because

$$\{ (a, d, t) \in \mathbb{R}^{2(n-1)} \times [1/2, 1] : (a + td, t) \in E^c \}$$

maps to $L^c$ under the orthogonal projection $(a, d, t) \rightarrow (a, d)$. Implies the decidability of the theory of real closed fields.

We can also suppose there is only one $a$ for each $d$.

Note there are many lines contained in many of the tubes of $\mathcal{T}_E$.

Indeed there is a $\delta$-ball of different directions $d$ for a single tube.

Thus, orthogonally projecting onto the directions $\Pi : (a, d) \mapsto d$,

$$\delta^{n-1} \# \mathcal{T}_E \leq C|\Pi(L)|.$$
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Application of Gromov’s algebraic lemma

We can piecewise parametrise the directions with polynomials;

\[ \Pi(L) = \bigcup_{j=1}^{N} G_{j}([0, 1]^{n-1}), \]

where the polys \( G_{j} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \) are of degree \( d \leq C(n, D) \).

We can also bound the number of pieces \( N \leq C(n, D) \).

Thus it will suffice to prove \( |G_{j}([0, 1]^{n-1})| \leq C(n, d)|E| \) for each \( j \).

In fact we consider a full parametrisation of \( L \) of the form

\[ L = \bigcup_{j=1}^{N} (F_{j}, G_{j})([0, 1]^{n-1}) \]

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($|I_x| = 1/4$)

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\[
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\((\text{FTA})\)

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\leq 4(8n)^n \int_{1/2}^1 |(F + tG)([0, 1]^{n-1})| \, dt \leq 4(8n)^n |E|,
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**Conjecture (Kakeya maximal)**

Let \( p > \frac{n}{n-1} = 1 + \frac{1}{n-1} \). Then there is a constant \( C \) such that

\[ \left\| \sum_{T \in \mathbb{T}} 1_T \right\|_p^p \leq C \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathbb{T} \right]. \tag{K_p} \]

\( (K_p) \) implies that the Hausdorff dimension of a Kakeya set is \( \geq p' \).

Córdoba (1977): \( p > 2 \)

Drury/Christ (1983/84): \( p \geq 1 + \frac{2}{n-1} \)

Bourgain (1991): \( p > 1 + \alpha(n), \quad \text{for an } \alpha(n) \in \left( \frac{2}{n}, \frac{2}{n-1} \right) \)

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We get an improvement in some intermediate dimensions

\[ \left\| \sum_{T \in \mathbb{T}} 1_T ^p \right\| _p \leq C \delta ^{n-(n-1)p} \left[ \delta ^{n-1} \# \mathbb{T} \right], \quad (K_p) \]

holds whenever \( p > 1 + \alpha_n \frac{1}{n-1} \) with

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha_n )</th>
<th>( \alpha_n )</th>
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<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>Córdoba</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{4}{3} + \varepsilon )</td>
<td>Katz-Zahl</td>
</tr>
<tr>
<td>4</td>
<td>1.437...</td>
<td>Katz-Zahl</td>
</tr>
<tr>
<td>5</td>
<td>( \left( \frac{5}{4} \right)^2 )</td>
<td>Hickman-R.</td>
</tr>
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<td>6</td>
<td>( \frac{5}{3} )</td>
<td>Wolff</td>
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<td>7</td>
<td>( \left( \frac{7}{6} \right)^3 )</td>
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</tr>
<tr>
<td>8</td>
<td>( \left( \frac{8}{7} \right)^4 )</td>
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<td>9</td>
<td>( \left( \frac{9}{8} \right)^4 )</td>
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<td>10</td>
<td>( \left( \frac{10}{9} \right)^5 )</td>
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<td>11</td>
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<td>( \left( \frac{12}{11} \right)^6 )</td>
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<td>13</td>
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</tr>
<tr>
<td>( \geq 98 )</td>
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Partitioning with the polynomial ham sandwich theorem

Here we consider the measure \( \mu(A) := \int_A \left[ \sum_{T \in \mathcal{T}} 1_T(x) \right]^p dx \).

**Theorem (Stone-Tukey)**

For any family of \( k \) subsets of \( \mathbb{R}^n \), there is a poly \( P_k : \mathbb{R}^n \rightarrow \mathbb{R} \) of degree \( \leq k^{1/n} \) and an associated variety \( \{ P_k = 0 \} \) that bisects each of the \( k \) subsets into two pieces of equal measure.

**Corollary**

Any subset of \( \mathbb{R}^n \) can be divided into \( 2^j \) subsets of equal measure by \( \{ P = 0 \} \) where \( P \) is of degree \( \leq 2^{j/n} \).
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A brief history of polynomials in Kakeya-type problems

Dvir (2008) proved Wolff’s finite field Kakeya conjecture.

Ellenberg-Oberlin-Tao (2009) proved, with lines $L \subset \mathbb{F}_q^n$,

$$\left\| \sum_{L \in \mathcal{L}} 1_L \right\|_p^p \leq Cq^{(n-1)p-n}[q \# \mathcal{L}], \quad p \geq \frac{n}{n-1}.$$ 

Guth (2010) proved the endpoint $n$-linear Kakeya inequality.


Bourgain-Guth (2011) proved $k$-linear Kakeya estimates.

Iliopoulos (2013) solved the joints problem in $\mathbb{R}^3$ with multiplicity.


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$$\left\| \sum_{L \in \mathbb{L}} 1_L \right\|_p^p \leq C q^{(n-1)p-n} [q \# \mathbb{L}], \quad p \geq \frac{n}{n-1}.$$  

Guth (2010) proved the endpoint $n$-linear Kakeya inequality.


Bourgain-Guth (2011) proved $k$-linear Kakeya estimates.

Iliopoulou (2013) solved the joints problem in $\mathbb{R}^3$ with multiplicity.


Du-Guth-Li (2017) proved Schrödinger maximal estimates in $\mathbb{R}^{2+1}$.

A brief history of polynomials in Kakeya-type problems

Dvir (2008) proved Wolff’s finite field Kakeya conjecture.
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\[
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\]

Guth (2010) proved the endpoint \( n \)-linear Kakeya inequality.
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Bourgain-Guth (2011) proved \( k \)-linear Kakeya estimates.
Iliopoulou (2013) solved the joints problem in \( \mathbb{R}^3 \) with multiplicity.

Guth-Katz (2015) solved the Erdős distance problem in \( \mathbb{R}^2 \).
Du-Guth-Li (2017) proved Schrödinger maximal estimates in \( \mathbb{R}^{2+1} \).

\[ \vdots \]
Sketch of the proof

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) $\left\| \sum_{T \in \mathbb{T}} 1_T \right\|^p_{L^p(\mathbb{R}^n)} = \sum_{O'} \left\| \sum_{T \in \mathbb{T}} 1_T \right\|^p_{L^p(O')}$, 

(ii) $O'$ denotes a piece of $\mathbb{R}^n \backslash \{ P = 0 \}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \backslash E$,

$\left\| \sum_{T \in \mathbb{T}} 1_T \right\|^p_{L^p(\mathbb{R}^n)} = \sum_{O} \left\| \sum_{T \in \mathbb{T}_O} 1_T \right\|^p_{L^p(O)} + \left\| \sum_{T \in \mathbb{T}_E} 1_T \right\|^p_{L^p(E)}$.

$\sum_{O} \# \mathbb{T}_O \leq d \# \mathbb{T}$ by the fundamental theorem of algebra (FTA),

$\Rightarrow \# \mathbb{T}_O \leq 4d^{-(n-1)} \# \mathbb{T}$ for most $O$.

$\# \mathbb{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i)  
$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(O')}^p,$$

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{P = 0\}$,

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If $E$ denotes the $\delta$-neighbourhood of $\{P = 0\}$ and $O := O' \setminus E$,

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p + \left\| \sum_{T \in \mathcal{T}_E} 1_T \right\|_{L^p(E)}^p.$$

$$\sum_{O} \#T_O \leq d\#T \quad \text{by the fundamental theorem of algebra (FTA),}$$

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For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

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(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{ P = 0 \}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \setminus E$,

\[ \| \sum_{T \in \mathcal{T}} 1_T \|^p_{L^p(\mathbb{R}^n)} = \sum_{O} \| \sum_{T \in \mathcal{T}_O} 1_T \|^p_{L^p(O)} + \| \sum_{T \in \mathcal{T}_E} 1_T \|^p_{L^p(E)} . \]

\[ \sum_O \# \mathcal{T}_O \leq d \# \mathcal{T} \quad \text{by the fundamental theorem of algebra (FTA)}, \]

\[ \Rightarrow \# \mathcal{T}_O \leq 4d^{-(n-1)} \# \mathcal{T} \quad \text{for most } O. \]

\# \mathcal{T}_E \quad \text{can be bounded using the polynomial Wolff axioms (PWA).} \]
Sketch of the proof

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) \[ \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(O')}^p, \]

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{ P = 0 \}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \setminus E$,

\[ \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p + \left\| \sum_{T \in \mathcal{T}_E} 1_T \right\|_{L^p(E)}^p. \]

\[ \sum_{O} \# \mathcal{T}_O \leq d \# \mathcal{T} \quad \text{by the fundamental theorem of algebra (FTA)}, \]

\[ \Rightarrow \# \mathcal{T}_O \leq 4d^{-(n-1)} \# \mathcal{T} \quad \text{for most } O. \]

$\# \mathcal{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) \[
\left\| \sum_{T \in \mathbb{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathbb{T}} 1_T \right\|_{L^p(O')}^p,
\]

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{ P = 0 \}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \setminus E$,

\[
\left\| \sum_{T \in \mathbb{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O} \left\| \sum_{T \in \mathbb{T}_O} 1_T \right\|_{L^p(O)}^p + \left\| \sum_{T \in \mathbb{T}_E} 1_T \right\|_{L^p(E)}^p.
\]

\[\sum_{O} \#\mathbb{T}_O \leq d \#\mathbb{T}\] by the fundamental theorem of algebra (FTA),

\[\Rightarrow \#\mathbb{T}_O \leq 4d^{-(n-1)} \#\mathbb{T}\] for most $O$.

$\#\mathbb{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) \[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left( \sum_{T \in \mathcal{T}} 1_T \right)_{L^p(O')}^p,
\]

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{P = 0\}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{P = 0\}$ and $O := O' \setminus E$,

\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O} \left( \sum_{T \in \mathcal{T}_O} 1_T \right)_{L^p(O)}^p + \left( \sum_{T \in \mathcal{T}_E} 1_T \right)_{L^p(E)}^p.
\]

\[
\sum_{O} \#\mathcal{T}_O \leq d \#\mathcal{T}
\]
by the fundamental theorem of algebra (FTA),

\[
\Rightarrow \#\mathcal{T}_O \leq 4d^{-(n-1)} \#\mathcal{T}
\]
for most $O$.

$\#\mathcal{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

1. \[ \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(O')}^p, \]

2. $O'$ denotes a piece of $\mathbb{R}^n \setminus \{ P = 0 \}$,

3. each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \setminus E$,

\[ \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p + \left\| \sum_{T \in \mathcal{T}_E} 1_T \right\|_{L^p(E)}^p. \]

\[ \sum_{O} \# \mathcal{T}_O \leq d \# \mathcal{T} \quad \text{by the fundamental theorem of algebra (FTA)}, \]

\[ \Rightarrow \# \mathcal{T}_O \leq 4d^{-(n-1)} \# \mathcal{T} \quad \text{for most } O. \]

$\# \mathcal{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof: nonalgebraic case

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) $\left\| \sum_{T \in T} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in T} 1_T \right\|_{L^p(O')}^p$,

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{ P = 0 \}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \setminus E$,

$\left\| \sum_{T \in T} 1_T \right\|_{L^p(\mathbb{R}^n)}^p \leq 4 \sum_{O} \left\| \sum_{T \in T_O} 1_T \right\|_{L^p(O)}^p$.

$\sum_{O} \# T_O \leq d \# T$ by the fundamental theorem of algebra (FTA),

$\Rightarrow \# T_O \leq 4d^{-(n-1)} \# T$ for most $O$.

$\# T_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof: nonalgebraic case

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i)  
\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(O')}^p,
\]

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{P = 0\}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{P = 0\}$ and $O := O' \setminus E$,

\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p \leq 8 \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p.
\]

\[
\sum_{O} \# \mathcal{T}_O \leq d \# \mathcal{T} \quad \text{by the fundamental theorem of algebra (FTA),}
\]

\[
\Rightarrow \# \mathcal{T}_O \leq 4d^{-(n-1)} \# \mathcal{T} \quad \text{for all } O.
\]

$\# \mathcal{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
Sketch of the proof: nonalgebraic case $k$ times

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) \[ \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(O')}^p, \]

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{P = 0\}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{P = 0\}$ and $O := O' \setminus E$,

\[ \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p. \]

\[ \sum_{O} \# \mathcal{T}_O \leq d^k \# \mathcal{T} \quad \text{by the fundamental theorem of algebra (FTA)}, \]

\[ \Rightarrow \# \mathcal{T}_O \leq 4d^{-(n-1)k} \# \mathcal{T} \quad \text{for all } O. \]

\[ \# \mathcal{T}_E \quad \text{can be bounded using the polynomial Wolff axioms (PWA)}. \]
Sketch of the proof: nonalgebraic case $k$ times

For any large $d$, we can find a poly $P$ of degree $\leq d$ such that

(i) $\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p = \sum_{O'} \left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(O')}^p$,

(ii) $O'$ denotes a piece of $\mathbb{R}^n \setminus \{ P = 0 \}$,

(iii) each summand is equal.

If $E$ denotes the $\delta$-neighbourhood of $\{ P = 0 \}$ and $O := O' \setminus E$,

$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_{L^p(\mathbb{R}^n)}^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p$.

$\sum_{O} \# \mathcal{T}_O \leq d^k \# \mathcal{T}$ by the fundamental theorem of algebra (FTA),

$\sup_{O} \# \mathcal{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)}$.

$\# \mathcal{T}_E$ can be bounded using the polynomial Wolff axioms (PWA).
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \sum_O \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p$$

$$\leq 8^k \sum_O \delta^n \left( \# \mathcal{T}_O \right)^p \leq 8^k \delta^n \left( \sup_O \# \mathcal{T}_O \right)^p \sum_O \# \mathcal{T}_O.$$

By (FTA),

$$\sup_O \# \mathcal{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_O \# \mathcal{T}_O \leq d^k \# \mathcal{T}.$$

Plugging this in, we get

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \delta^n \left[ 4d^{-(n-1)k} \delta^{-(n-1)} \right]^{p-1} d^k \# \mathcal{T}$$

$$= 4^{p-1} \left[ 8d^{n-(n-1)p} \right]^{k} \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right]$$

$$\leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right]$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p \leq 8^k \sum_{O} \delta^n [\# \mathcal{T}_O]^p \leq 8^k \delta^n [\sup_{O} \# \mathcal{T}_O]^{p-1} \sum_{O} \# \mathcal{T}_O.$$  

By (FTA),

$$\sup_{O} \# \mathcal{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)}$$ and $$\sum_{O} \# \mathcal{T}_O \leq d^k \# \mathcal{T}.$$  

Plugging this in, we get

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \delta^n [4d^{-(n-1)k} \delta^{-(n-1)}]^{p-1} d^k \# \mathcal{T} \leq 4^{p-1} \left[8d^{n-(n-1)p}\right]^k \delta^{n-(n-1)p} \left[\delta^{n-1} \# \mathcal{T}\right] \leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \left[\delta^{n-1} \# \mathcal{T}\right]$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}$$

$$\leq 8^k \sum_{O} \delta^n \left[ \# \mathcal{T}_O \right]^p \leq 8^k \delta^n \left[ \sup_{O} \# \mathcal{T}_O \right]^{p-1} \sum_{O} \# \mathcal{T}_O.$$

By (FTA),

$$\sup_{O} \# \mathcal{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_{O} \# \mathcal{T}_O \leq d^k \# \mathcal{T}.$$

Plugging this in, we get

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \delta^n \left[ 4d^{-(n-1)k} \delta^{-(n-1)} \right]^{p-1} d^k \# \mathcal{T}$$

$$= 4^{p-1} \left[ 8d^{n-(n-1)p} \right]^k \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right]$$

$$\leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right]$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in \mathbb{T}} 1_T \right\|_p^p \leq 8^k \sum_{O \in \mathbb{T}_O} \left\| \sum_{T \in \mathbb{T}_O} 1_T \right\|_{L^p(O)}^p \leq 8^k \sum_{O} \delta^n \left[ \#\mathbb{T}_O \right]^p \leq 8^k \delta^n \left[ \sup_{O} \#\mathbb{T}_O \right]^{p-1} \sum_{O} \#\mathbb{T}_O.$$

By (FTA),

$$\sup_{O} \#\mathbb{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_{O} \#\mathbb{T}_O \leq d^k \#\mathbb{T}.$$

Plugging this in, we get

$$\left\| \sum_{T \in \mathbb{T}} 1_T \right\|_p^p \leq 8^k \delta^n \left[ 4d^{-(n-1)k} \delta^{-(n-1)} \right]^{p-1} d^k \#\mathbb{T} \leq 4^{p-1} \left[ 8d^{n-(n-1)p} \right]^k \delta^{n-(n-1)p} \left[ \delta^{n-1} \#\mathbb{T} \right] \leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \left[ \delta^{n-1} \#\mathbb{T} \right],$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in T} 1_T \right\|_p^p \leq 8^k \sum_{O} \left\| \sum_{T \in T_O} 1_T \right\|_{L_p(O)}^p$$

$$\leq 8^k \sum_{O} \delta^n \left[ \#T_O \right]^p \leq 8^k \delta^n \left[ \sup_{O} \#T_O \right]^{p-1} \sum_{O} \#T_O.$$

By (FTA),

$$\sup_{O} \#T_O \leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_{O} \#T_O \leq d^k \#T.$$

Plugging this in, we get

$$\left\| \sum_{T \in T} 1_T \right\|_p^p \leq 8^k \delta^n \left[ 4d^{-(n-1)k} \delta^{-(n-1)} \right]^{p-1} d^k \#T$$

$$= 4^{p-1} \left[ 8d^{n-(n-1)p} \right]^k \delta^{n-(n-1)p} \left[ \delta^{n-1} \#T \right]$$

$$\leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \left[ \delta^{n-1} \#T \right]$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p \leq 8^k \sum_{O} \delta^n \left[ \# \mathcal{T}_O \right]^p \leq 8^k \delta^n \left[ \sup_{O} \left\# \mathcal{T}_O \right] \right]^p \sum_{O} \# \mathcal{T}_O.$$

By (FTA),

$$\sup_{O} \left\# \mathcal{T}_O \right\leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_{O} \# \mathcal{T}_O \leq d^k \# \mathcal{T}.$$

Plugging this in, we get

$$\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \delta^n \left[ 4d^{-(n-1)k} \delta^{-(n-1)} \right]^{p-1} d^k \# \mathcal{T} \leq 4^{p-1} \left[ 8d^{n-(n-1)p} \right]^{k} \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right] \leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right]$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If $|O| \leq \delta^n$ after $k$ steps:

$$\left\| \sum_{T \in \mathbb{T}} 1_T \right\|_p^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathbb{T}_O} 1_T \right\|_{L^p(O)}^p \leq 8^k \sum_{O} \delta^n [\#\mathbb{T}_O]^p \leq 8^k \delta^n [\sup_{O} \#\mathbb{T}_O]^{p-1} \sum_{O} \#\mathbb{T}_O.$$  

By (FTA),

$$\sup_{O} \#\mathbb{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_{O} \#\mathbb{T}_O \leq d^k \#\mathbb{T}.$$  

Plugging this in, we get

$$\left\| \sum_{T \in \mathbb{T}} 1_T \right\|_p^p \leq 8^k \delta^n [4d^{-(n-1)k} \delta^{-(n-1)}]^{p-1} d^k \#\mathbb{T} \leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} [\delta^{n-1} \#\mathbb{T}]$$

if $p > \frac{n}{n-1}$ and $d$ is chosen sufficiently large at the beginning.
If \(|O| \leq \delta^n\) after \(k\) steps:

\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \sum_O \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p \\
\leq 8^k \sum_O \delta^n [\# \mathcal{T}_O]^p \leq 8^k \delta^n [\sup_O \# \mathcal{T}_O]^{p-1} \sum_O \# \mathcal{T}_O.
\]

By (FTA),

\[
\sup_O \# \mathcal{T}_O \leq 4d^{-(n-1)k} \delta^{-(n-1)} \quad \text{and} \quad \sum_O \# \mathcal{T}_O \leq d^k \# \mathcal{T}.
\]

Plugging this in, we get

\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \delta^n [4d^{-(n-1)k} \delta^{-(n-1)}]^{p-1} \sum_O \# \mathcal{T}_O \\
= 4^{p-1} [8d^{n-(n-1)p}]^k \delta^{n-(n-1)p} \sum_O \# \mathcal{T}_O \\
\leq 4^{p-1} \times 1 \times \delta^{n-(n-1)p} \sum_O \# \mathcal{T}_O
\]

if \(p > \frac{n}{n-1}\) and \(d\) is chosen sufficiently large at the beginning. \(\square\)
If $|O| \leq \delta^n$ after $k$ steps in the $m$-dimensional case:

\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \sum_{O} \left\| \sum_{T \in \mathcal{T}_O} 1_T \right\|_{L^p(O)}^p \\
\leq 8^k \sum_{O} \delta^n (\# \mathcal{T}_O)^p \leq 8^k \delta^n \left( \sup_{O} \# \mathcal{T}_O \right)^p \sum_{O} \# \mathcal{T}_O.
\]

By (FTA) and (PWA),

\[
\sup_{O} \# \mathcal{T}_O \leq 4d^{-(m-1)k} \delta^{-(m-1)} \quad \text{and} \quad \sum_{O} \# \mathcal{T}_O \leq d^k \# \mathcal{T}.
\]

Plugging this in, we get

\[
\left\| \sum_{T \in \mathcal{T}} 1_T \right\|_p^p \leq 8^k \delta^n \left[ 4d^{-(m-1)k} \delta^{-(m-1)} \right]^{p-1} d^k \# \mathcal{T}
\]

\[
= 4^{p-1} \left[ 8d^{m-(m-1)p} \right]^k \delta^{m-(m-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right] \\
\leq 4^{p-1} \times 1 \times \delta^{m-(m-1)p} \left[ \delta^{n-1} \# \mathcal{T} \right]
\]

if $p > \frac{m}{m-1}$ and $d$ is chosen sufficiently large at the beginning. □
A strengthened algebraic result

Before I told you about

Theorem (Katz-R.)

For all \( D \in \mathbb{N} \), there is a constant \( C \) such that

\[
\delta^{n-1} |T_E| \leq C |E|
\]

whenever \( E \subset \mathbb{R}^n \) is a semialgebraic set of complexity \( \leq D \).

Moreover the following is true:

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For all \( D \in \mathbb{N} \), there is a constant \( C \) such that

\[
\delta^{n-1} \# \{ T \in \mathcal{T} : |T \cap E| \geq \lambda |T| \} \leq C \lambda^{-n} |E|
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happy bday Andreas!