Estimates for the resolvent of the Laplacian

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Madison Lectures in Fourier Analysis in honor of Andreas Seeger

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Resolvent estimate for the Laplacian

• For a given an operator $T$ the resolvent is formally given by

$$(T - zI)^{-1}, \; z \in \mathbb{C},$$

and used to understanding the spectrum of $T$.

• We consider $T = -\Delta$ on $\mathbb{R}^d$. 

Applications to related problems:

▶ uniform Sobolev estimate
▶ unique continuation
▶ limiting absorption principle
▶ absolute continuity of the spectrum of the Schrödinger operators
▶ eigenvalue bounds for the Schrödinger operators
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• We consider $T = -\Delta$ on $\mathbb{R}^d$. For $z \in \mathbb{C} \setminus [0, \infty)$

$$( -\Delta - z )^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \frac{1}{|\xi|^2 - z} \hat{f}(\xi) d\xi.$$
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- We are interested in the estimate with $C = C(z, p, q)$
  
  $$\|(-\Delta - z)^{-1}f\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \ \forall z \in \mathbb{C} \setminus [0, \infty).$$
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- Applications to related problems:
  - uniform Sobolev estimate
  - unique continuation
  - limiting absorption principle
  - absolute continuity of the spectrum of the Schrödinger operators
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Uniform Sobolev inequality

• For $1 \leq k \leq d$ let $Q$ be the non-degenerate quadratic form given by

\[ Q(\xi) = \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_d^2. \]

• For $a = (a_1, \ldots, a_j, \ldots) \in \mathbb{C}^d$ and $b \in \mathbb{C}$,
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• For $a = (a_1, \ldots, a_j, \ldots) \in \mathbb{C}^d$ and $b \in \mathbb{C}$, let $P$ be a second order differential operator defined by

$$P(a, b, D) = Q(D) + a \cdot \nabla + b, \quad D = -i\nabla.$$
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Theorem (Kenig-Ruiz-Sogge, ’87)

Let \(d \geq 3\) and \(Q(D) = -\Delta\). Then

\[
\|u\|_{L^q(\mathbb{R}^d)} \leq C\|P(a, b, D)u\|_{L^p(\mathbb{R}^d)}, \quad u \in S(\mathbb{R}^d)
\]

holds with \(C\) independent of \(a_j\) and \(b\) if and only if

\[
\frac{1}{p} - \frac{1}{q} = \frac{2}{d} \quad \text{and} \quad \frac{2d}{d + 3} < p < \frac{2d}{d + 1}.
\]
When $Q(D) \neq -\Delta$
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\frac{1}{p} - \frac{1}{q} = \frac{2}{d} \quad \text{and} \quad \frac{2d(d - 1)}{d^2 + 2d - 1} < p < \frac{2(d - 1)}{d}.
\]

Furthermore, if \( p = \frac{2(d-1)}{d} \) or \( \frac{2d(d-1)}{d^2+2d-1} \), we have

\[
\| u \|_{q, \infty} \leq C \| P(a, b, D)u \|_{p,1}.
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When $Q(D) \neq -\Delta$

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Furthermore, if $p = \frac{2(d-1)}{d}$ or $\frac{2d(d-1)}{d^2 + 2d - 1}$, we have $\|u\|_{q,\infty} \leq C \|P(a, b, D)u\|_{p,1}$.

- The range is smaller than that of the case $Q(D) = -\Delta$. This is due to the fact that the surface $\xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_d^2 = 0$ has $d - 1$ non-vanishing principal curvatures.
Application

Corollary (Carleman estimate)

Let $d \geq 3$ and let $p$, $q$, and $P(a, b, D)$ be as in the above theorems, then we have

$$\|e^{-v \cdot x} u\|_{L^q(\mathbb{R}^d)} \leq C \|e^{-v \cdot x} P(a, b, D) u\|_{L^p(\mathbb{R}^d)},$$

where $C$ is independent of vectors $v \in \mathbb{R}^d$, and $a_j, b \in \mathbb{C}$.

• Replacing $u$ with $e^{-v \cdot x} u$, the estimate is equivalent to

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C \|P(a, b, D+i v) u\|_{L^p(\mathbb{R}^d)},$$

$u \in S(\mathbb{R}^d)$.

• The estimates have applications to unique continuation property.

Corollary (Unique continuation)

Let $p$ and let $P(a, b, D)$ be as in the above theorem and $V \in L^d/2(\mathbb{R}^d)$. Suppose $u \in W^{2, p}(\mathbb{R}^d)$, $u$ is supported in a half space, and $|P(a, b, D) u| \leq |V u|$. Then $u = 0$ on the whole space $\mathbb{R}^d$. 

Sanghyuk Lee (Seoul National University Madison Lectures in Fourier Analysis in honor of Andreas Seeger)
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where \( C \) is independent of vectors \( \mathbf{v} \in \mathbb{R}^d, \) and \( a_j, b \in \mathbb{C}. \)
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In particular with $a = 0$ and $b = z$, the uniform estimate equals

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$$\|(Q(D) - z)^{-1}u\|_{L^q(\mathbb{R}^d)} \leq C\|u\|_{L^p(\mathbb{R}^d)}.$$ 

• Taking $z = s \pm i\epsilon$, $s, \epsilon > 0$,

$$\frac{1}{Q(\xi) - s - i\epsilon} - \frac{1}{Q(\xi) - s + i\epsilon} = \frac{2i\epsilon}{(Q(\xi) - s)^2 + \epsilon^2} \rightarrow 2\pi i \delta(Q(\xi) - s)$$

as $\epsilon \rightarrow 0$ in the sense of distribution.
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• Uniform Sobolev inequality $\Rightarrow$ restriction-extension estimate (to and from) the surface $\{\xi : Q(\xi) = s\}$:
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- Uniform Sobolev inequality \( \Rightarrow \) restriction-extension estimate (to and from) the surface \( \{ \xi : Q(\xi) = s \} \):

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\| \int \delta(Q(\xi) - s)e^{2\pi ix \cdot \xi} \hat{f}(\xi) d\xi \|_{L^q(\mathbb{R}^d)} \leq C \| f \|_{L^p(\mathbb{R}^d)}, \quad \forall f \in S(\mathbb{R}^d).
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- When \( q = p' \), the estimates were typically shown by imbedding the operator in a suitable analytic family of operators.
Back to the resolvent of the Laplacian

• For \( z \in \mathbb{C} \setminus [0, \infty) \) let us set

\[
\|(−Δ − z)^{-1}\|_{p→q} := \inf \left\{ B : \|(-Δ − z)^{-1}f\|_{L^q(\mathbb{R}^d)} \leq B\|f\|_{L^p(\mathbb{R}^d)}, \forall f \in \mathcal{S}(\mathbb{R}^d) \right\}.
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• **Goal:** Find the precise value of $\|(−Δ − z)^{-1}\|_{p→q}$ up to a multiplicative constant.
For $z \in \mathbb{C} \setminus [0, \infty)$ let us set

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**Goal:** Find the precise value of $\|(-\Delta - z)^{-1}\|_{p \to q}$ up to a multiplicative constant. In other words, we want to find $\mathcal{K} : \mathbb{C} \setminus [0, \infty) \to [0, \infty)$ such that

$$C^{-1}\mathcal{K}(z) \leq \|(-\Delta - z)^{-1}\|_{p \to q} \leq C\mathcal{K}(z).$$
Back to the resolvent of the Laplacian

- For $z \in \mathbb{C} \setminus [0, \infty)$ let us set
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    $$C^{-1} \mathcal{K}(z) \leq \|(−\Delta − z)^{-1}\|_{p→q} \leq C \mathcal{K}(z).$$

  - Thanks to homogeneity (and scaling $\xi \to |z|\xi$),
    
    $$\|(−\Delta − z)^{-1}\|_{p→q} = |z|^{-1+\frac{d}{2}(\frac{1}{p}−\frac{1}{q})} \left\| \left( −\Delta − \frac{Z}{|Z|} \right)^{-1} \right\|_{p→q}, \forall z \in \mathbb{C} \setminus [0, \infty).$$
Back to the resolvent of the Laplacian

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• Enough to consider
  \[
  \|(-\Delta - z)^{-1}\|_{p \to q}, \quad z \in S^1 \setminus \{1\}.
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Uniformly boundedness range with $z \in S^1 \setminus \{1\}$

Theorem (Kenig-Ruiz-Sogge '87, Gutiérrez '04)

Let $d \geq 2$, $1 \leq p, q \leq \infty$, $z \in S^1 \setminus \{1\}$. Then $\|(-\Delta - z)^{-1}\|_{p \to q} \leq C$ holds with $C$ independent of $z$ if and only if

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- $L^{p,1}-L^{q,\infty}$ estimate (restricted weak type) holds if $(1/p, 1/q)$ is either

$$B(d) := \left(\frac{d+1}{2d}, \frac{(d-1)^2}{2d(d+1)}\right), \text{ or } B'(d) := \left(\frac{d^2 + 4d - 1}{2d(d+1)}, \frac{d-1}{2d}\right).$$
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- Failure for \( (1/p, 1/q) \notin \mathcal{R}_1 \) was already known in the study of the Bochner–Riesz operators of negative orders.
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Admissible range of $p, q$

**Proposition (Basic region)**

Let $d \geq 2$, $z \in \mathbb{C} \setminus [0, \infty)$, and $1 \leq p, q \leq \infty$. Then $\|(-\Delta - z)^{-1}\|_{p \to q} < \infty$
Admissible range of $p, q$

**Proposition (Basic region)**

Let $d \geq 2$, $z \in \mathbb{C} \setminus [0, \infty)$, and $1 \leq p, q \leq \infty$. Then $\|(-\Delta - z)^{-1}\|_{p \rightarrow q} < \infty$ if and only if $(1/p, 1/q)$ is contained in

$$\mathcal{R}_0(d) := \begin{cases} 
\{(a, b) \in [0, 1]^2 : 0 \leq a - b < 1\} & \text{if } d = 2, \\
\{(a, b) \in [0, 1]^2 : 0 \leq a - b \leq \frac{2}{d}\} \setminus \{(1, \frac{d-2}{d}), (\frac{2}{d}, 0)\} & \text{if } d \geq 3.
\end{cases}$$
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(a, b) \in [0, 1]^2 : 0 \leq a - b \leq \frac{2}{d} & \text{if } d \geq 3.
\end{cases}$$

- Behaves like the fractional integration operator. In fact,

$$\partial_x^\alpha ((|\xi|^2 - z)^{-1} |\xi|^2) = O(|\xi|^{-|\alpha|}).$$

- If $p \neq 1$ and $q \neq \infty$ and $0 \leq 1/p - 1/q \leq 2/d$, by the Mikhlin’s multiplier theorem and the Hardy-Littlewood-Sobolev inequality.

- If $p = 1$ or $q = \infty$, handled differently.
Look at the multiplier

• For a given bounded measurable function $m$ we define

$$m(D)f = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} m(\xi)\hat{f}(\xi)e^{ix\cdot\xi} d\xi.$$
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• With $z = a + bi \in \mathbb{S} \setminus \{1\}$,

$$\mathcal{R}^z(\xi) := \frac{1}{|\xi|^2 - z} = \frac{|\xi|^2 - a + ib}{(|\xi|^2 - a)^2 + b^2}.$$ 

Natural to expect that the bound gets worse as $|b| \to 0$. 

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Look at the multiplier

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- If $|b| > \delta_0 > 0$ or $|b| \leq \delta_0$ and $a < 0$, $\mathcal{R}^z$ is smooth multiplier near the origin and behaves like $|\xi|^{-2}$ at infinity.
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$$\|(−\Delta - z)^{-1}\|_{p\to q} = \|\mathcal{R}^z(D)\|_{p\to q} \leq C.$$
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• As $b \to 0$ $R^z$ becomes singular and in the sense of distribution

$$\lim_{b \to 0} R^z(\xi) = P.V. \frac{1}{|\xi|^2 - 1} + 2\pi i\delta(|\xi|^2 - 1).$$
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Proposition

Let $d \geq 2$, $z \in S^1 \setminus \{1\}$, and $1 \leq p, q \leq \infty$. Then

$$\|(-\Delta - z)^{-1}\|_{p \to q} \gtrsim \text{dist}(z, [0, \infty))^{-\gamma(p,q)},$$
Lower bounds

**Proposition**

Let $d \geq 2$, $z \in S^1 \setminus \{1\}$, and $1 \leq p, q \leq \infty$. Then

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where

$$\gamma(p, q) := \max \left\{ 0, 1 − \frac{d + 1}{2} \left( \frac{1}{p} − \frac{1}{q} \right), \frac{d + 1}{2} − \frac{d}{p}, \frac{d}{q} − \frac{d − 1}{2} \right\}.$$

- Since $\|\mathcal{R}^z(D)\|_{p→q} = \|\mathcal{R}^z(D)\|_{p→q}$, $\|(\mathcal{R}^z ± \overline{\mathcal{R}^z})(D)\|_{p→q} \leq 2\|\mathcal{R}^z(D)\|_{p→q}$. 
Lower bounds

**Proposition**

Let $d \geq 2$, $z \in S^1 \setminus \{1\}$, and $1 \leq p, q \leq \infty$. Then

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- Since $\|\mathcal{R}^z(D)\|_{p \to q} = \|\overline{\mathcal{R}^z(D)}\|_{p \to q}$, $\|(\mathcal{R}^z \pm \overline{\mathcal{R}^z})(D)\|_{p \to q} \leq 2\|\mathcal{R}^z(D)\|_{p \to q}$.
- We may work with the imaginary part

$$\frac{1}{2i} \left( \mathcal{R}^z - \overline{\mathcal{R}^z} \right) = \frac{b}{(|\xi|^2 - a)^2 + b^2}.$$
• Taking $0 < b \ll 1$ and $a = 1$, we need to show $\exists f$ (depending on $b, p, q$) such that

$$
\left\| \mathcal{F}^{-1}\left(\frac{b \hat{f}(\xi)}{(|\xi|^2 - 1)^2 + b^2}\right)\right\|_q \gtrsim \max \left\{ 1, b^{-1 + \frac{d+1}{2}} \left(\frac{1}{p} - \frac{1}{q}\right), b^{\frac{d-1}{2}} - \frac{d}{q}, b^\frac{d}{p} - \frac{d+1}{2} \right\} \| f \|_p.
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• **Knapp type example:** With small enough $k$ and $c$, we set

$$\hat{f}(\xi) = \psi \left( \frac{\xi_d - 1}{kb} \right) \prod_{j=1}^{d-1} \psi \left( \frac{\xi_j}{c \sqrt{b}} \right).$$
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Roughly, $\mathcal{F}^{-1} \left( \frac{b \hat{f}(\xi)}{(|\xi|^2 - 1)^2 + b^2} \right) \sim b^{-1} f$ and $\| f \|_r \sim a^{\frac{d+1}{2}(1 - \frac{1}{r})}$. 
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• **Concentration near the sphere** $S^{d-1}$: We take $\hat{f} = \chi_0(|\xi|)$ and $\chi_0$ is supported in $[1 - \epsilon_0, 1 + \epsilon_0]$. 
Taking $0 < b \ll 1$ and $a = 1$, we need to show $\exists f$ (depending on $b, p, q$) such that

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**Concentration near the sphere $S^{d-1}$:** We take $\hat{f} = \chi\circ(|\xi|)$ and $\chi\circ$ is supported in $[1 - \epsilon_0, 1 + \epsilon_0]$. If $|x| \sim b^{-1}$, then

$$\left| \mathcal{F}^{-1}\left( \frac{b \chi\circ(|\xi|)}{(|\xi|^2 - 1)^2 + b^2} \right)(x) \right| \sim |\hat{d}\sigma(x)| \sim |x|^{-\frac{d-1}{2}} \sim b^{\frac{d-1}{2}}.$$
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**Knapp type example:** With small enough $k$ and $c$, we set

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Roughly, $\mathcal{F}^{-1}\left(\frac{b \hat{f}(\xi)}{(|\xi|^2 - 1)^2 + b^2}\right) \sim b^{-1} f$ and $\| f \|_r \sim a^{d+1} (1 - \frac{1}{r})$.

**Concentration near the sphere $S^{d-1}$:** We take $\hat{f} = \chi_\circ(|\xi|)$ and $\chi_\circ$ is supported in $[1 - \epsilon_0, 1 + \epsilon_0]$. If $|x| \sim b^{-1}$, then

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$$

So, $\| \mathcal{F}^{-1}\left(\frac{b \chi_\circ(|\xi|)}{(|\xi|^2 - 1)^2 + b^2}\right) \|_q \gtrsim b^{\frac{d-1}{2} - \frac{d}{q}}$ while $\| f \|_p \sim 1$. 
Conjecture

Since $\text{dist}(\frac{z}{|z|}, [0, \infty)) = |z|^{-1} \text{dist}(z, [0, \infty))$, it is natural to conjecture the following:

**Conjecture (Sharp resolvent estimate)**

For $d \geq 2$ and $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_0(d) \setminus \{\text{some endpoint cases}\}$, the bound

$$\|(-\Delta - z)^{-1}\|_{p \to q} \sim |z|^{-1 + \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) + \gamma(p,q)} \text{dist}(z, [0, \infty))^{-\gamma(p,q)}.$$

holds with the implicit constant independent of $z \in \mathbb{C} \setminus [0, \infty)$. 
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holds with the implicit constant independent of $z \in \mathbb{C} \setminus [0, \infty)$.

- It turns out the problem is closely related to the Bochner-Riesz problem, especially of negative order.
• Recall $\gamma(p, q) := \max \left\{ 0, 1 - \frac{d+1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \frac{d+1}{2} - \frac{d}{p}, \frac{d}{q} - \frac{d-1}{2} \right\}$. 

• $d \geq 3$

• The value $\gamma(p, q)$ divides $R_0$ regions into four regions $R_1, R_2, R_3, R'_3$.

• $H_0 = \left( \frac{1}{2}, \frac{1}{2} \right)$

• $D_0 = \left( d - \frac{1}{2}, d - \frac{1}{2} \right)$

• If $X = (a, b)$, $X' = (1 - b, 1 - a)$.

• $B = \left( d + \frac{1}{2}, \left( d - 1 \right) \frac{1}{2} \frac{d}{2} \right)$

• $A = \left( d + \frac{1}{2}, d - \frac{3}{2} \right)$

• Figure: The case $d \geq 3$. 

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• Recall \( \gamma(p, q) := \max \left\{ 0, 1 - \frac{d+1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \frac{d+1}{2} - \frac{d}{p}, \frac{d}{q} - \frac{d-1}{2} \right\} \).

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\[ \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}'_3 \]
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**Figure:** The case \( d \geq 3 \).
Recall $\gamma(p, q) := \max\left\{ 0, 1 - \frac{d+1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \frac{d+1}{2} - \frac{d}{p}, \frac{d}{q} - \frac{d-1}{2} \right\}$.

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**Figure:** The case $d \geq 3$. 

Sanghyuk Lee (Seoul National University Madison Lectures in Fourier Analysis in honor of Andreas Seeger) 

Estimates for the resolvent of the Laplacian 

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**Figure:** The case $d \geq 3$. 
• Recall \( \gamma(p, q) := \max \left\{ 0, \ 1 - \frac{d+1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \ \frac{d+1}{2} - \frac{d}{p}, \ \frac{d}{q} - \frac{d-1}{2} \right\} \).

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Figure: The case \( d \geq 3 \).
Figure: The case $d = 2$. 
Main result

Theorem (Kwon–L.’18)

If \( d = 2 \) the conjecture is true. For \( d \geq 3 \), the conjecture is true if \( \left( \frac{1}{p}, \frac{1}{q} \right) \) is contained in the shaded region except the dashed lines.
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If \( d = 2 \) the conjecture is true. For \( d \geq 3 \), the conjecture is true if \((\frac{1}{p}, \frac{1}{q})\) is contained in the shaded region except the dashed lines.

- The \( L^p - L^q \) boundedness fails if \((\frac{1}{p}, \frac{1}{q}) \in [B, A] \cup [B', A']\). But restricted weak type and weak type bounds holds.

\[ \frac{1}{q} \]

\[ \frac{1}{p} \]

\[ \frac{2}{d} \]

\[ \frac{d+1}{2d} \]

\[ \frac{d}{d+1} \]
Main result

**Theorem (Kwon–L.’18)**

If \( d = 2 \) the conjecture is true. For \( d \geq 3 \), the conjecture is true if \( \left( \frac{1}{p}, \frac{1}{q} \right) \) is contained in the shaded region except the dashed lines.

- The \( L^p - L^q \) boundedness fails if \( \left( \frac{1}{p}, \frac{1}{q} \right) \in [B, A] \cup [B', A'] \). But restricted weak type and weak type bounds holds.
- The \( L^p - L^q \) bounds holds with the extra factor

\[
\log |z| - \log \text{dist}(z, [0, \infty)) + 1
\]
Numerology

• Let \( P_{\ast} = \left( \frac{1}{p_{\ast}}, \frac{1}{p_{\ast}} \right) \) with

\[
\frac{1}{p_{\ast}} := \begin{cases} 
\frac{3(d-1)}{2(3d+1)} & \text{if } d \text{ is odd} \\
\frac{3d-2}{2(3d+2)} & \text{if } d \text{ is even}
\end{cases}
\]

• \( P_{\circ} = \left( \frac{1}{p_{\circ}}, \frac{1}{q_{\circ}} \right) \) where

\[
\left( \frac{1}{p_{\circ}}, \frac{1}{q_{\circ}} \right) := \begin{cases} 
\left( \frac{(d+5)(d-1)}{2(d^2+4d-1)}, \frac{(d-1)(d+3)}{2(d^2+4d-1)} \right) & \text{if } d \text{ is odd} \\
\left( \frac{d^2+3d-6}{2(d^2+3d-2)}, \frac{(d-1)(d+2)}{2(d^2+3d-2)} \right) & \text{if } d \text{ is even}
\end{cases}
\]
Location of complex eigenvalues of $-\Delta + V$

Spectral region: For $p$, $q$, $d$ in the theorem and $\ell > 0$ we set $Z_{p, q}^{\ell} := \{ z \in \mathbb{C} \setminus [0, \infty) : \| (\Delta - z) - 1 \|_{p \to q} \leq \ell \}$

Corollary: Let $p$, $q$ be as in the main theorem and let $C > 0$ be the implicit constant for the upper bound. Suppose that $\| V \|_{L^{pq}} \leq \frac{1}{2C\ell}$. If $E \in C \setminus [0, \infty)$ and $u \in L^{q}(\mathbb{R}^d) \{ 0 \}$ satisfy $(\Delta - V)u = Eu$, then $E \in Z_{p, q}^{\ell}$.

Proof: If $E$ were in $Z_{p, q}^{\ell}$, then $\| (\Delta - E) - 1 \|_{p \to q} \leq C\ell$. $\| u \|_{q} \leq C\ell \| (\Delta - E)u \|_{p} \leq C\ell \| (\Delta - V - E)u \|_{p} + \| Vu \|_{p} = C\ell \| Vu \|_{p} \leq C\ell \| V \|_{pq} \| u \|_{q} \leq \frac{1}{2\ell} \| u \|_{q}$, which implies $u \equiv 0$. Contradiction!
Location of complex eigenvalues of $-\Delta + V$

- Spectral region: For $p, q, d$ in the theorem and $\ell > 0$ we set

$$Z_{p,q}(\ell) := \left\{ z \in \mathbb{C} \setminus [0, \infty) : \|(-\Delta - z)^{-1}\|_{p \to q} \leq \ell \right\}$$

$$\subset \left\{ z \in \mathbb{C} \setminus [0, \infty) : |z|^{-1 + \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) + \gamma(p,q)} \text{dist}(z, [0, \infty))^{-\gamma(p,q)} \lesssim \ell \right\}$$
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**Corollary**

Let $p, q$ be as in the main theorem and let $C > 0$ be the implicit constant for the upper bound. Suppose that $\|V\|_{L^\frac{pq}{q-p} (\mathbb{R}^d)} \leq 1/(2C\ell)$. 
Location of complex eigenvalues of $-\Delta + V$

- Spectral region: For $p, q, d$ in the theorem and $\ell > 0$ we set
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  \[
  \subset \left\{ z \in \mathbb{C} \setminus [0, \infty) : |z|^{-1 + \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) + \gamma(p,q)} \text{dist}(z, [0, \infty)) - \gamma(p,q) \lesssim \ell \right\}
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Let $p, q$ be as in the main theorem and let $C > 0$ be the implicit constant for the upper bound. Suppose that $\|V\|_{L^\frac{pq}{q-p}(\mathbb{R}^d)} \leq 1/(2C\ell)$. If $E \in \mathbb{C} \setminus [0, \infty)$ and $u \in L^q(\mathbb{R}^d) \setminus \{0\}$ satisfy $(-\Delta + V)u = Eu$, which implies $u \equiv 0$. Contradiction!
Location of complex eigenvalues of $-\Delta + V$

- **Spectral region:** For $p, q, d$ in the theorem and $\ell > 0$ we set
  \[
  \mathcal{Z}_{p,q}(\ell) := \left\{ z \in \mathbb{C} \setminus [0, \infty) : \|( -\Delta - z )^{-1} \|_{p \to q} \leq \ell \right\}
  \subset \left\{ z \in \mathbb{C} \setminus [0, \infty) : |z|^{-1 + \frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \gamma(p,q) \text{dist}(z, [0, \infty))} \right\}^{-\gamma(p,q)} \lesssim \ell
  \]

**Corollary**

Let $p, q$ be as in the main theorem and let $C > 0$ be the implicit constant for the upper bound. Suppose that $\| V \|_{L^{\frac{pq}{q-p}}(\mathbb{R}^d)} \leq 1/(2C\ell)$. If $E \in \mathbb{C} \setminus [0, \infty)$ and $u \in L^q(\mathbb{R}^d) \setminus \{0\}$ satisfy $(-\Delta + V)u = Eu$, then $E \in \mathbb{C} \setminus \mathcal{Z}_{p,q}(\ell)$.

- **Proof.** If $E$ were in $\mathcal{Z}_{p,q}(\ell)$, then $\|( -\Delta - E )^{-1} \|_{p \to q} \leq C\ell$.
  \[
  \| u \|_q \leq C\ell \|( -\Delta - E )u \|_p \leq C\ell \left( \| ( -\Delta + V - E )u \|_p + \| Vu \|_p \right)
  = C\ell \| Vu \|_p \leq C\ell \| V \|_{\frac{pq}{q-p}} \| u \|_q \leq \frac{1}{2} \| u \|_q,
  \]
  which implies $u \equiv 0$. Contradiction!
Spectral regions

- If \( \left( \frac{1}{p}, \frac{1}{q} \right) \in \mathcal{R}_1 \setminus ([B, A] \cup [A, A'] \cup [A', b']) \)
• If $p, q$ are as in Theorem and $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_2$, 

![Graph showing a region in the complex plane with points indicated.]

• Im $E \to 0$ as $|E| \to \infty$. 
If $p, q$ are as in Theorem and $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_3 \cap \Delta ABD$, 

\[ \begin{array}{c}
0 \quad \frac{2}{d} \quad \frac{d+1}{2d} \quad 1 \quad \frac{1}{p} \\
\frac{1}{q} \end{array} \]
• If $p, q$ are as in Theorem and $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_3 \cap \triangle ABD$,

![Diagram](image1)

• If $p, q$ are as in Theorem and $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_3 \setminus ABD$,

![Diagram](image2)
On compact Riemannian manifolds

- Let \((M, g)\) be a \(d\)-dimensional compact Riemannian manifold without boundary.
On compact Riemannian manifolds

• Let \((M, g)\) be a \(d\)-dimensional compact Riemannian manifold without boundary.

**Theorem (Dos Santos Ferreira, Kenig, and Salo, 14’)**

Let \(d \geq 3\). For \(z \in \Xi_\delta := \{ z \in \mathbb{C} \setminus [0, \infty) : \text{Im} \sqrt{z} \geq \delta \}\) there is a uniform constant \(C\) independent of \(z\) such that

\[
\| (-\Delta_g - z)^{-1} f \|_{L^{\frac{2d}{d-2}}(M)} \leq C \| f \|_{L^{\frac{2d}{d+2}}(M)}.
\]

• This resembles the shape of spectral region for \((\frac{1}{p}, \frac{1}{q}) \in \tilde{R}_3, \tilde{R}_3'\).
On compact Riemannian manifolds

- Let \((M, g)\) be a \(d\)-dimensional compact Riemannian manifold without boundary.

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- Spectral regions for which the uniform \( L^{\frac{2d}{d+2}} - L^{\frac{2d}{d-2}} \) resolvent estimate holds.

**Figure:** Compact manifold case

- This resembles the shape of spectral region for \( \left( \frac{1}{p}, \frac{1}{q} \right) \in \tilde{R}_3, \tilde{R}'_3 \).
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Figure: Compact manifold case

Figure: Euclidean case

- This resembles the shape of spectral region for \((\frac{1}{p}, \frac{1}{q}) \in \widetilde{\mathcal{R}}_3, \widetilde{\mathcal{R}}'_3\).
Bochner–Riesz operator of negative order

\[ B_\alpha f = F^{-1}(\hat{f}(\xi)^{\alpha} + \Gamma(1 + \alpha)\hat{f}(\xi)) \]

for \( \alpha > -1 \) and for \( \alpha \leq -1 \) it is defined by analytic continuation.

For \( d \geq 2 \) and \( \alpha \in (0, \frac{d+1}{2}) \)

\[ P_\alpha(d) := \{(a, b) \in [0, 1]^2 : a - b \geq \frac{2\alpha d + 1}{2d + \alpha d}, a > \frac{d - 1}{2d + \alpha d}, b < \frac{d + 1}{2d - \alpha d}\} \]

Conjecture

Let \( d \geq 2 \) and \( 0 < \alpha < \frac{d+1}{2} \).

\( B_\alpha \) is bounded from \( L^p \) to \( L^q \) if and only if \( (1/p, 1/q) \in P_\alpha(d) \).
Bochner–Riesz operator of negative order

Let $\Gamma$ is the gamma function. The classical Bochner–Riesz operator is given by

$$B^\alpha f = \mathcal{F}^{-1}\left( \frac{(1 - |\xi|^2)^\alpha}{\Gamma(1 + \alpha)} \hat{f}(\xi) \right)$$

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- 
  \[
  \mathcal{F}^{-1}(\hat{f} d\sigma)(x) = \frac{1}{(2\pi)^d} \int_{S^{d-1}} \hat{f}(\theta) e^{ix \cdot \theta} d\sigma(\theta) = B^{-1} f(x),
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- 

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- For $d \geq 2$ and $\alpha \in (0, \frac{d+1}{2})$

$$\mathcal{P}_\alpha(d) := \left\{ (a, b) \in [0,1]^2 : a - b \geq \frac{2\alpha}{d+1}, a > \frac{d-1}{2d} + \frac{\alpha}{d}, b < \frac{d+1}{2d} - \frac{\alpha}{d} \right\}.$$
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Conjecture

Let $d \geq 2$ and $0 < \alpha < \frac{d+1}{2}$. $B^{-\alpha}$ is bounded from $L^p$ to $L^q$ if and only if $(1/p, 1/q) \in P_\alpha(d)$. 

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Sanghyuk Lee (Seoul National University Madison) Estimation for the resolvent of the Laplacian May 16, 2019 23 / 31
Earlier results were obtained by Börjeson ('86), Sogge ('86), Carbery-Soria ('88), Seeger ('88).
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Partial results regarding the critical estimate \( \left( \frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{d+1} \right) \) were obtained by Bak, McMichael and D. Oberlin ('95).

When \( d = 2 \), the problem was settled by Bak ('97).

In higher dimensions the conjecture was verified by Cho, Kim, L. and Shim ('02) for
\[
\alpha > \frac{(d-2)(d+1)}{2(d-1)(d+2)}
\]
relying on the bilinear restriction estimates due to Tao.

Theorem (Kwon-L. '18)

Let \( d \geq 3 \). If
\[
\alpha > \begin{cases} 
\frac{(d+1)(d-1)}{2(d+2)} & \text{when } d \text{ is odd}, \\
\frac{(d+1)(d-2)}{2(d+3)} & \text{when } d \text{ is even},
\end{cases}
\]
then the conjecture is true.
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**Theorem (Kwon-L. ’18)**

Let \( d \geq 3 \). If \( \alpha > \frac{(d+1)(d-1)}{2(d^2+4d-1)} \) when \( d \) is odd, and \( \alpha > \frac{(d+1)(d-2)}{2(d^2+3d-2)} \) when \( d \) is even, then the conjecture is true.
Brief overview on proof

- We may assume \( z = 1 + i\delta \) and \( 0 < |\delta| \leq \epsilon_0 \) for fixed \( 0 < \epsilon_0 \ll 1 \).
- The associated multiplier

\[
m_\delta(\xi) = \frac{1}{|\xi|^2 - z} \chi\left(\frac{|\xi| - 1}{\epsilon_0}\right) = \frac{|\xi|^2 - 1 + i\delta}{(|\xi|^2 - 1)^2 + \delta^2} \chi\left(\frac{|\xi| - 1}{\epsilon_0}\right).
\]

- Need to show

\[
\|m_\delta(D)f\|_q \lesssim \delta^{-\gamma(p,q)} \|f\|_p, \quad 0 < \delta \ll 1.
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Brief overview on proof

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$$\|m_\delta(D)f\|_q \lesssim \delta^{-\gamma(p,q)} \|f\|_p, \quad 0 < \delta \ll 1.$$  

• Slowly decaying multiplier

$$\frac{|\xi|^2 - 1 + i\delta}{(|\xi|^2 - 1)^2 + \delta^2} = \sum_{i=1}^{2} \frac{1}{\delta} \psi_i\left(\frac{1 - |\xi|^2}{\delta}\right), \quad \psi_1(t) = \frac{t}{t^2 + 1}, \quad \psi_1(t) = \frac{i}{t^2 + 1}.$$
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• We may assume $z = 1 + i\delta$ and $0 < |\delta| \leq \epsilon_0$ for fixed $0 < \epsilon_0 \ll 1$.

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• Break $m_\delta(\xi)$ dyadically away the sphere $|\xi| = 1$ with $\beta \in C^\infty_c([-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2])$ so that

$$\frac{1}{\delta} \psi_i\left(\frac{1 - |\xi|^2}{\delta}\right) = \delta \sum_{\lambda: \text{dayadic} \geq 1} \psi_i\left(\frac{1 - |\xi|^2}{\delta}\right) \beta\left(\frac{1 - |\xi|^2}{\lambda\delta}\right).$$
Generalization

- We need to generalize the multiplier operator to include those operators which appear via finite decomposition and recaling.

- We say $\psi \in \mathbb{E}ll(N, \epsilon)$ if
  - $\psi : [-1, 1]^{d-1} \to \mathbb{R}$, $\psi(0) = 0$ and $\nabla \psi(0) = 0$;
  - $\sup_{\xi' \in [-1,1]^{d-1}} \max_{0 \leq |\alpha| \leq N} \left| \partial_\xi^\alpha \psi(\xi') - \frac{1}{2} |\xi'|^2 \right| \leq \epsilon$.

- For $\psi \in \mathbb{E}ll(N, \epsilon)$ define
  \[
  \mathcal{M}_{\delta, \lambda}(\xi) := \varphi\left( \frac{m(\xi)(\xi_d - \psi(\xi'))}{\delta} \right) \beta\left( \frac{m(\xi)(\xi_d - \psi(\xi'))}{\delta \lambda} \right) \chi_0(\xi)
  \]
  - $m$ smooth function with $m \sim 1$ on the support of the cutoff function.
  - $\chi_0 \in C_\infty(\mathbb{R}^d)$ is supported in a small nbd of the origin,
  - $\varphi \in C^\infty(\mathbb{R})$ such that $|\varphi^{(k)}(t)| \lesssim (1 + |t|)^{-k-1}$ for $k = 0, 1, 2, \cdots$. 
Key estimate

Proposition (Kwon–L. ’18)

Let $0 < \delta \ll 1$ and $\lambda \geq 1$. For $p, q$ satisfying $\frac{1}{q} = \frac{d-1}{d+1} (1 - \frac{1}{p})$ and

$$\frac{d}{2(d+2)} \frac{1}{p} < \frac{1}{p} < \frac{1}{2},$$


\[
\left\| M_{\delta, \lambda}(D) \hat{f}(\xi) \right\|_{L^q(\mathbb{R}^d)} \leq C \lambda^{-1}(\delta \lambda) \frac{d}{p} \frac{d-1}{2} \| f \|_{L^p(\mathbb{R}^d)}.
\]

- Follow strategy used for the negative Bochner-Riesz operator by Cho-Kim-L.-Shim
- Interpolation between linear and bilinear estimates give bilinear estimates on wider range.
- Deducing linear estimates from the bilinear estimates
Bilinear and linear estimates

Theorem

Let $q > \frac{2(d+2)}{d}$, $a_\circ \in (2^{-5}, 1/2]$ and $\psi \in \mathcal{E}(N, \epsilon)$. Suppose that

$$(\xi', \xi_d) \in \text{supp} \hat{f}_1, \quad (\zeta', \zeta_d) \in \text{supp} \hat{f}_2 \quad \implies \quad |\xi' - \zeta'| \geq a_\circ.$$ 

Then there is a constant $C$, independent of $\delta, \lambda$ and $\psi$, such that

$$\left\| \frac{1}{\lambda} \prod_{k=1,2} m_{\delta, \lambda}(D) f_k \right\|_{L^{q/2}(\mathbb{R}^d)} \leq C \delta \lambda^{-1} \prod_{k=1,2} \| f_k \|_{L^2(\mathbb{R}^d)}.$$ 

This is derived from a variant of Tao’s bilinear restriction estimate.
Let
\[ \frac{1}{p^*_v} := \begin{cases} \frac{3(d-1)}{2(3d+1)} & \text{if } d \text{ is odd} \\ \frac{3d-2}{2(3d+2)} & \text{if } d \text{ is even} \end{cases} \]

Theorem

Let \( p^*_v < p \leq \infty \). Then there exist a large \( N > 0 \), a small \( \epsilon > 0 \) and a constant \( C > 0 \) such that

\[
\| M_{\delta, \lambda} (D)f \|_{L^p(\mathbb{R}^d)} \leq C \lambda^{-1}(\delta \lambda)^{d-\frac{d-1}{2}} \| f \|_{L^p(\mathbb{R}^d)},
\]

where the constants \( C \) are independent of \( \delta, \lambda, \psi \in E\|I(N, \epsilon) \).

- Use so called the Carleson-Sjölin type reduction argument which is based on the stationary phase method and asymptotic expansion. This produces the oscillatory kernel.
- To obtain sharp bound for the oscillatory convolution operator, we use the recent result due to Guth, Hickman (when \( d \geq 4 \)) and Iliopoulou who showed sharp oscillatory integral estimate under the addition elliptic condition.
The fractional Laplacian \((-\Delta)^{\frac{s}{2}}, 0 < s < d\)
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- The resolvent for the fractional Laplacian \((-\Delta)^{\frac{s}{2}}\):

\[
((-\Delta)^{\frac{s}{2}} - z)^{-1} f(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} (|\xi|^s - z)^{-1} \hat{f}(\xi) d\xi.
\]
The fractional Laplacian \((-\Delta)^{s/2}, 0 < s < d\)

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  \[
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  \]

- Uniform bounds on \(\|\left((-\Delta)^{s/2} - z\right)^{-1}\|_{p \rightarrow q}\) with \(p, q = p'\) on a certain range were obtained by Cuenin to study eigenvalues of the fractional Schrödinger operators with complex potentials.
The fractional Laplacian \((-\Delta)^{\frac{s}{2}}\), 0 < s < d

• The resolvent for the fractional Laplacian \((-\Delta)^{\frac{s}{2}}\):

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• Uniform bounds on \(\|((-\Delta)^{\frac{s}{2}} - z)^{-1}\|_{p\to q}\) with \(p, q = p'\) on a certain range were obtained by Cuenin to study eigenvalues of the fractional Schrödinger operators with complex potentials.

• Later, uniform bounds up to the optimal range of \(p, q\) were obtained by Huang, Yao, and Zheng for \(\frac{2d}{d+1} \leq s < d\).
The fractional Laplacian \((-Δ)^{\frac{s}{2}}, 0 < s < d\)

- The resolvent for the fractional Laplacian \((-Δ)^{\frac{s}{2}}\):

\[
((-Δ)^{\frac{s}{2}} - z)^{-1} f(ξ) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iξ \cdot x} (|ξ|^s - z)^{-1} \hat{f}(ξ) dξ.
\]

- Uniform bounds on \(\|((-Δ)^{\frac{s}{2}} - z)^{-1}\|_{p→q}\) with \(p, q = p'\) on a certain range were obtained by Cuenin to study eigenvalues of the fractional Schrödinger operators with complex potentials.

- Later, uniform bounds up to the optimal range of \(p, q\) were obtained by Huang, Yao, and Zheng for \(\frac{2d}{d+1} \leq s < d\).

- By the same argument we also have the similar result for the fractional Laplacian except some endpoint cases:

\[
\|((-Δ)^{\frac{s}{2}} - z)^{-1}\|_{p→q} ∼ |z|^{-1+\frac{d}{s}(\frac{1}{p} - \frac{1}{q}) + \gamma(p, q)} \text{dist}(z, [0, ∞))^{-\gamma(p, q)}.
\]
Congratulations!