MATH848-TOPICS IN NUMBER THEORY—SHIMURA VARIETIES AND AUTOMORPHIC FORMS

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In this topics course, we talk about generalization of the classical modular curves and modular forms, which is about GL_2 theory, to high dimensional analogues induced by Shimura in 60s and reformulated by Deligne in early 70s. They are now called Shimura varieties. It is very interesting just like modular curves as it has both rich geometry structure and a lot of symmetries given by the underline (reductive) groups. It is thus a combination of geometry and representation theory (or harmonic analysis if you like) and you can study it from different points of view.

Instead of covering general theory, we will focus on some examples (unitary Shimura varieties, also Siegel modular varieties and/or orthogonal Shimura varieties) to understand at least one case better. To me, it is also easier as it is more concrete and does not need tons of background. Each one of them is already very rich in research. Taking unitary Shimura variety of type $U(n,1)$ for example (the one I am mostly interested in right now), we will discuss

(1) The associated complex manifold X.
(2) Its special cycles of codimension r for each $r \leq n$ (sub-Shimura varieties).
(3) Its canonical compactification, and its boundary.
(4) Why it is actually defined over a number field? (moduli interpretation).
(5) Certainly automorphic forms as sections of some tautological line bundle on X.
(6) Construction of such automorphic forms with divisors supported on special divisors (Borcherds liftings).
(7) ....

By the way, the Shimura varieties for $U(1,1)$ are cousins of modular curves. The ones for $U(2,1)$ are Picard modular varieties, known well before Shimura.

Unfortunately there is no text book to follow. We will go with notes and some papers.

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James Milne's online notes 'Introduction to Shimura varieties' will be our basic reference for basic concepts on Shimura varieties.

http://www.jmilne.org/math/Books/AA1988a.pdf

There are a few advanced proceedings on this topic which can be good reference once when you would like to know more about a specific subject. One of them is two books)

http://www.jmilne.org/math/Books/AA1988a.pdf

http://www.jmilne.org/math/Books/AA1988b.pdf good references

A conference proceeding 'Picard modular surfaces' edited by Langlands and Ramakrishnan contains important stuff about Picard modular surfaces, while Rogawski's research book ' unitary automophic forms of 3 variables' contains a lot of stuff about automorphic forms on $U(2,1)$. although we will not go that deep in detail. A little warning is that both books are not easy to read.

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