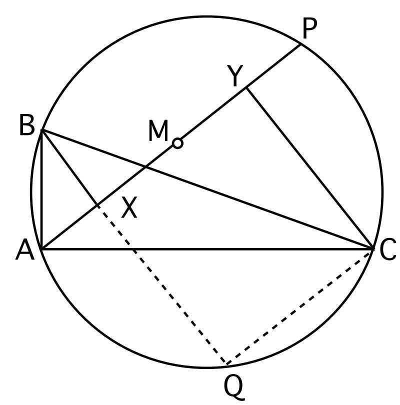
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET IV (2009-2010)

1. Recall that most cubic polynomials have either three distinct real roots or just one such root. Sometimes, however, a cubic has exactly two distinct real roots because two of the three roots coincide. Find all real numbers a such that the polynomial $x^3 - 7x^2 + ax - 9$ has exactly two distinct real roots.

SOLUTION. Suppose the polynomial $x^3 - 7x^2 + ax - 9$ has a repeated root r and some other root s. It is easy to see that the coefficient -7 of x^2 is the negative of the sum of the three roots and that the constant term -9 is the negative of the product of the three roots. Thus $2r + s = 7$ and $r^2 s = 9$, so $s = 7 - 2r$ and $9 = r^2 s = r^2(7 - 2r)$. This yields the equation $2r^3 - 7r^2 + 9 = 0$, and while it is often difficult to solve a cubic equation, it is easy to see that $r = -1$ is a solution of this one. Thus we can factor $2r^3 - 7r^2 + 9 = (r + 1)P(r)$ for some polynomial $P(r)$. We deduce by long division of polynomials that $P(r) = 2r^2 - 9r + 9$, so to find the other possible values for r, we must solve the quadratic equation $2r^2 - 9r + 9 = 0$. Either by factoring or else by using the quadratic formula, we find the two solutions $r = 3$ and $r = 3/2$, and thus the three possibilities for r are -1, 3 and $3/2$, and the corresponding values for $s = 7 - 2r$ are 9, 1 and 4. In each of these cases, the original cubic has exactly two distinct roots. To find the corresponding values of a, we use the fact that the coefficient a of x is the sum of each of the three products of two of the roots, so $a = r^2 + 2rs = r^2 + 2r(7 - 2r) = 14r - 3r^2$. If $r = -1$, we see that $a = -14 - 3 = -17$; if $r = 3$, we have $a = 42 - 27 = 15$, and if $r = 3/2$, we have $a = 21 - 27/4 = 57/4$.

2. In the diagram, $\angle BAC = 90°$. The bisector of this angle is extended to meet the circumcircle of $\triangle ABC$ at point P. Show that $AB + AC = \sqrt{2}\,AP$.



SOLUTION. Drop perpendiculars $\overline{BX}$ and $\overline{CY}$ from B and C to $\overline{AP}$, as shown. Observe that $\triangle ABX$ is a $45°, 45°, 90°$ triangle, and thus $AB = \sqrt{2}\,AX$, and similarly $AC = \sqrt{2}\,AY$. It suffices, therefore, to show that $AX + AY = AP$, or equivalently, that $AX = YP$. Now extend $\overline{BX}$ to meet the circle at Q and draw $\overline{QC}$. Since $\angle BAC = 90°$, we see that $\overline{BC}$ is a diameter of the circle, and thus $\angle Q = 90°$, and thus $XQCY$ is a rectangle. Since $\overline{QC}$ and $\overline{AP}$ are parallel chords, they have a common perpendicular bisector (which is a diameter of the circle). Because $XQCY$ is a rectangle, this bisector also bisects $\overline{XY}$, and thus the midpoint M of $\overline{XY}$ is also the midpoint of $\overline{AP}$. It follows that $AX = AM - XM = PM - YM = YP$, as wanted.

3. Suppose that 100 on-off light switches are mounted on a control panel in one long line, and that I can flip any batch of consecutive switches simultaneously. Some of the lights are on and some are off, and I want to turn them all off. For example, if lights 1 through 30 are on; lights 31 through 60 are off and lights 61 through 100 are on, I can turn all the lights off in two moves: first flip switches 1 through 30 and then flip switches 61 through 100. Find the smallest number N so that I never need more than N moves to turn off all the lights.

SOLUTION. The answer is $N = 50$. To see this, we define an "on-island" to be a consecutive set of switches that are turned on, where this set is not contained in any larger set of consecutive on-switches. Thus in the example given in the statement of the problem, switches 1 through 30 form an on-island and switches 61 through 100 form another on-island. Given two on-islands, there must be at least one off-switch between them, so if there are a total of m on-islands, there must be at least $m-1$ off-switches separating them. Since each of the on-islands contains at least one switch, the total number of switches is at least $m+(m-1)$, and thus $2m-1 \leq 100$, and we deduce that $m \leq 50$. Since all of the switches in an on-island can be turned off simultaneously, I can turn off all the lights in at most 50 moves, working one on-island at a time.

To see that no number of moves smaller than 50 is always sufficient, we define a "jump" to be a pair of consecutive switches, where one is on and the other is off. Flipping a consecutive set of switches can increase or decrease the number of jumps, but the change can never be more than two jumps because a jump can be created or destroyed only at the ends of the set of flipped switches. Now consider the case where the even-numbered switches are all on and the odd-numbered switches are all off. There are 99 jumps here, and since each move can decrease this number by at most 2, we see that 49 or fewer moves will leave at least one jump, and so not all the lights will be off. In this even-odd case, therefore, it requires 50 moves to turn off all the lights.

4. Find all solutions in real numbers x, y and z for the equations

$$x^2+y^2+z^2 = xy+xz+yz = xyz.$$

SOLUTION. We have

$$0 = 2(x^2+y^2+z^2-xy-xz-yz) = (x-y)^2+(x-z)^2+(y-z)^2\,.$$

Since each of the three squares on the right of the above equation is nonnegative, the only way their sum can be zero is if each of them is zero. We conclude that $x=y=z$. Now $x^3 = xyz = x^2+y^2+z^2 = 3x^2$. If x is nonzero, we can divide through by x^2 to deduce that $x=3$. It follows that the only solutions to the given equations are $x=y=z=0$ and $x=y=z=3$.

5. Does there exist a positive integer n for which it is possible to write $1/n = 1/a^2+1/b^2$, where a and b are unequal positive integers? If so, find the smallest such number n.

SOLUTION. First, observe that $1/10^2+1/5^2 = 1/100+1/25 = 5/100 = 1/20$, so we can take $n=20$ and write $1/n = 1/a^2+1/b^2$ for different integers a and b. We argue that n cannot be smaller than 20. To see thus, let d be the largest integer that divides both a and b, and write $a=dx$ and $b=dy$. Observe that x and y cannot have a common prime factor because if p were such a prime, then dp would be a common factor of a and b exceeding d. Multiplying both sides of the equation $1/a^2+1/b^2 = 1/n$ by na^2, we get $n+na^2/b^2 = a^2$, and thus na^2/b^2 is an integer equal to nx^2/y^2. Since no prime divisor of y divides x, it follows that n/y^2 is an integer and similarly, n/x^2 is an integer. In other words, both x^2 and y^2 divide n, and again using the fact that x and y have no common prime factor, we deduce that n is a multiple of x^2y^2.

Since a and b are unequal, we can assume that $a<b$, and thus $x<y$. Assuming that $n \leq 20$, we consider the possibilities for x and y such that n is a multiple of x^2y^2. If $x=1$, then y is one of 2, 3 or 4, and if $x>1$, then x is at least 2 and y is at least 3, and hence x^2y^2 is at least 36, which is too big since we are assuming that $n \leq 20$. We conclude that $x=1$ and that y is one of 2, 3 or 4. Continuing to assume that $n \leq 20$, we have $a=d$ and $b=yd$, and thus $1/n = 1/d^2+1/y^2d^2 = (y^2+1)/y^2d^2$. Thus $n = y^2d^2/(y^2+1)$. If $y=2$, then $n=4d^2/5$ so d must be a multiple of 5 and the smallest possibility is $n=20$, which occurs when $d=5$. If $y=3$, we get $n=9d^2/10$, so d must be a multiple of 10, and the smallest possibility in this case is $n=90$, which, of course, exceeds 20. Finally, if $y=4$, we get $n=16d^2/17$, so d must be a multiple of 17, and the smallest possibility in this case is $n=272$, which also exceeds 20. This proves that n cannot be smaller than 20.