1. Let $K$ be a field, and let $f(x) \in K[x]$ be an irreducible polynomial. Suppose that the splitting field $L$ of $f(x)$ is a Galois extension of $K$ (that is, $f$ is separable). Let $\alpha \in L$ be one of the roots of $f$.
   (a) Show that if $\text{Gal}(L/K)$ is an abelian group, then $L = K(\alpha)$.
   (b) Is the converse statement true? That is, suppose $L = K(\alpha)$; must $\text{Gal}(L/K)$ be abelian?

2. Let $A$ be a real skew-symmetric $n \times n$ matrix such that $\text{im}(A) = \text{ker}(A)$. In particular, $\dim(\text{im}(A)) = n/2$, so $n$ must be even.
   (a) Let $V$ be an $n/2$-dimensional subspace of $\mathbb{R}^n$. Define a bilinear form $(,)$ on $V$ by
      $$(v, w) = \langle v, Aw \rangle;$$
      here $\langle , \rangle$ is the standard Euclidean product on $\mathbb{R}^n$. Show that $(,)$ is a skew-symmetric form on $V$.
   (b) Show that $V$ can be chosen so that the above form $(,)$ is non-degenerate.
   (c) Show that $n$ must be divisible by 4. If you use some facts about non-degenerate skew-symmetric forms, sketch their proofs.

3. Let $\phi : A \to B$ be a homomorphism of commutative rings. Recall that $\phi$ is said to be integral if every element of $b \in B$ is integral over $\phi(A)$, and that $\phi$ is finite if $B$ is a finitely generated $A$-module.
   (a) Give an example of a map of commutative rings $A \to B$ that is integral but not finite.
   (b) Prove the Lying Over Theorem: Let $\phi : A \to B$ be an integral map of commutative rings. If $p \subset A$ is any prime ideal, then there exists a prime ideal $q \subset B$ such that $\phi^{-1}(q) = p$.

4. Let $D_k$ be the dihedral group of order $2k$, where $k \geq 3$.
   (a) Show that the number of automorphisms of the group $D_k$ is equal to $k \cdot \phi(k)$. Here $\phi$ is Euler’s $\phi$-function.
   (b) The automorphisms of $D_k$ form a group; let us denote it by $\text{Aut}(D_k)$. What is the structure of $\text{Aut}(D_k)$? Describe the group as explicitly as you can.

5. Consider the group $\text{GL}_n(\mathbb{Q})$ of invertible $n \times n$ matrices with rational coefficients. Suppose $G \subset \text{GL}_n(\mathbb{Q})$ is a finite subgroup. Prove that every prime factor $p$ of the order $|G|$ satisfies $p \leq n + 1$. 