1. For this problem (and this problem only) your answer will be graded on correctness alone, and no justification is necessary. Give an example of:
   (a) A group $G$ with a normal subgroup $N$ such that $G$ is not a semidirect product $N \rtimes G/N$.
   (b) A finite group $G$ that is nilpotent but not abelian.
   (c) A group $G$ whose commutator subgroup $[G,G]$ is equal to $G$.
   (d) A non-cyclic group $G$ such that all Sylow subgroups of $G$ are cyclic.
   (e) A transitive action of $S_3$ on a set $X$ of cardinality greater than 3.

2. Let $n > 0$ be an integer. Let $F$ be a field of characteristic 0, let $V$ be a vector space over $F$ of dimension $n$, and let $T : V \to V$ be an invertible $F$-linear map such that $T^{-1} = T$.
   Denote by $W$ the vector space of linear transformations from $V$ to $V$ that commute with $T$. Find a formula for $\dim(W)$ in terms of $n$ and the trace of $T$.

3. Let $R$ be a commutative ring with unity. Show that a polynomial $f(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_0 \in \mathbb{R}[t]$ is nilpotent if and only if all of its coefficients $c_0, \ldots, c_n \in R$ are nilpotent.

4. This is a question about “biquadratic extensions,” in two parts.
   (a) Let $F/E$ be a degree-4 Galois extension, where $E$ and $F$ are fields of characteristic different from 2. Show that $\text{Gal}(F/E) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ if and only if there exist $x, y \in E$ such that $F = E(\sqrt{x}, \sqrt{y})$ and none of $x, y, xy$ are squares in $E$.
   (b) Give an example of a field $E$ of characteristic 2 that is not algebraically closed but that has no Galois extension $F/E$ with Galois group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

5. Consider the ring $R = \mathbb{C}[x]$.
   (a) Describe all simple $R$-modules.
   (b) Give an example of an $R$-module that is indecomposable, but not simple. (Recall that a module is indecomposable if it cannot be written as a direct sum of non-trivial submodules.)
   (c) Consider $R$-modules $M = R/(x^3 + x^2)$ and $N = R/(x^3)$, and take their tensor product over $R$: $M \otimes_R N$. It is an $R$-module, and in particular, a vector space over $\mathbb{C}$. What is its dimension over $\mathbb{C}$?
   (d) Let $M$ be any $R$-module such that $\dim_\mathbb{C} M < \infty$, and let $N = R/(x^3)$, as before. Show that $\dim_\mathbb{C}(M \otimes_R N) = \dim_\mathbb{C} \text{Hom}_R(N, M)$. 