ALGEBRA QUALIFYING EXAM, JANUARY 2017

1. For this problem (and this problem only) your answer will be graded on correctness alone, and no justification is necessary. Give an example of:

(a) A group G with a normal subgroup N such that G is not a semidirect product $N \rtimes G/N$.
(b) A finite group G that is nilpotent but not abelian.
(c) A group G whose commutator subgroup $[G, G]$ is equal to G.
(d) A non-cyclic group G such that all Sylow subgroups of G are cyclic.
(e) A transitive action of S_3 on a set X of cardinality greater than 3.

2. Let $n > 0$ be an integer. Let F be a field of characteristic 0, let V be a vector space over F of dimension n, and let $T : V \to V$ be an invertible F-linear map such that $T^{-1} = T$.

Denote by W the vector space of linear transformations from V to V that commute with T. Find a formula for $\dim(W)$ in terms of n and the trace of T.

3. Let R be a commutative ring with unity. Show that a polynomial

$$f(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_0 \in R[t]$$

is nilpotent if and only if all of its coefficients $c_0, \ldots, c_n \in R$ are nilpotent.

4. This is a question about "biquadratic extensions," in two parts.

(a) Let F/E be a degree-4 Galois extension, where E and F are fields of characteristic different from 2. Show that $\mathrm{Gal}(F/E) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ if and only if there exist $x, y \in E$ such that $F = E(\sqrt{x}, \sqrt{y})$ and none of x, y, xy are squares in E.
(b) Give an example of a field E of characteristic 2 that is not algebraically closed but that has no Galois extension F/E with Galois group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

5. Consider the ring $R = \mathbb{C}[x]$.

(a) Describe all simple R-modules.
(b) Give an example of an R-module that is indecomposable, but not simple. (Recall that a module is *indecomposable* if it cannot be written as a direct sum of non-trivial submodules.)
(c) Consider R-modules $M = R/(x^3 + x^2)$ and $N = R/(x^3)$, and take their tensor product over R: $M \otimes_R N$. It is an R-module, and in particular, a vector space over $\mathbb{C}$. What is its dimension over $\mathbb{C}$?
(d) Let M be any R-module such that $\dim_{\mathbb{C}} M < \infty$, and let $N = R/(x^3)$, as before. Show that

$$\dim_{\mathbb{C}}(M \otimes_R N) = \dim_{\mathbb{C}} \mathrm{Hom}_R(N, M).$$