I.2.3: The key idea here is what they mean by the inequality in each case. In \( a \), it is inequalities of numbers, in \( b \) it is inequalities of functions, and in \( c \) it is ambiguous. None of the inequalities are true.

By direct computation, the integral in \( a \) is 
\[
\left[ x - \frac{x^3}{3} \right]_2^4 = (4 - \frac{64}{3}) - (2 - \frac{8}{3}) = -\frac{50}{3}.
\]
Alternatively, we could use the fact that \( 1 - x^2 \) is always negative on the interval \([2, 4]\), so the integral in \( a \) must be negative, so the inequality in \( a \) is false.

For part \( b \), note that we are integrating \( (1 - x^2) \) with respect to \( t \), from 2 to 4. Therefore, the integral is \((1 - x^2)(4 - 2) = 2(1 - x^2)\). As a function, this is not always positive, so the inequality in \( b \) is false.

For part \( c \), we are finding \( \int (1 + x^2) \, dx = x + \frac{x^3}{3} + C \). However, this is not a well-defined function, as it depends on what \( C \) we pick. Therefore, it doesn’t make sense to ask if \( x + \frac{x^3}{3} + C > 0 \), so this inequality can’t be true.

I.7.5: Let \( I_n = \int (\sin x)^n \, dx \). From equation 6.3, we have the formula:
\[
I_n = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} I_{n-2}
\]

We need to use this formula to compute \( \int \sin^2(x) \, dx \). Using the formula, we get:
\[
\int \sin^2(x) \, dx = I_2 = -\frac{1}{2} \sin^1(x) \cos(x) + \frac{1}{2} I_0
\]
\[
= -\frac{1}{2} \sin(x) \cos(x) + \frac{1}{2} \int \sin^0(x) \, dx
\]
\[
= -\frac{1}{2} \sin(x) \cos(x) + \frac{1}{2} x + C
\]

Alternatively, we can integrate \( \sin^2(x) \) using the half angle formula as follows:
\[
\int \sin^2(x) \, dx = \int \frac{1}{2} (1 - \cos(2x)) \, dx
\]
\[
= \frac{1}{2} \int (1 - \cos(2x)) \, dx
\]
\[
= \frac{1}{2} x - \frac{1}{4} \sin(2x) + C
\]
\[
= \frac{1}{2} x - \frac{1}{2} \sin(x) \cos(x) + C
\]

The last step was using the fact that \( \sin(2x) = 2 \sin(x) \cos(x) \). We get the same answer as before.
I.7.20: Let $I_n = \int \frac{dx}{(1+x^2)^n}$. If you look at the example in 6.4, you will find that as long as $n \neq 0$, the reduction formula still works. In the case where $n = 0$, this becomes $\int \frac{dx}{x}$, which is just $x + C$. Letting $n = -\frac{1}{2}$, we get:

$$\int \sqrt{1 + x^2} \, dx = \int (1 + x^2)^{1/2} \, dx = \int \frac{dx}{(1 + x^2)^{-1/2}} = I_{-1/2}$$

Additionally, we have:

$$\int \frac{dx}{\sqrt{1 + x^2}} = \int \frac{dx}{(1 + x^2)^{1/2}} = I_{1/2}$$

So, using the reduction formula to compare $I_{1/2}$ and $I_{-1/2}$, we have:

$$I_{1/2} = I_{-1/2} + \frac{1}{2(-1/2)} \frac{x}{(1 + x^2)^{-1/2}} + \frac{2(-1/2) - 1}{2(-1/2)} I_{-1/2}$$

$$\rightarrow I_{1/2} = -x \sqrt{1 + x^2} + 2I_{-1/2}$$

I.7.21: We use integration by parts to integrate $\frac{1}{x}$. Let $F = \frac{1}{x}, G' = 1$. Then $F' = -\frac{1}{x^2}, G = x$. So, we get:

$$\int \frac{1}{x} \, dx = FG - \int F'G \, dx$$

$$= \frac{1}{x} - \int \frac{-1}{x^2} \, dx$$

$$= 1 + \int \frac{1}{x} \, dx$$

Subtracting, we get $\int \frac{1}{x} \, dx - \int \frac{1}{x} \, dx = 1$. This is not wrong though! Remember that antiderivatives can be different up to a constant. So, let $c_1$ be the constant for the first integral, and $c_2$ be the constant for the second.

Then $\int \frac{1}{x} \, dx - \int \frac{1}{x} \, dx = c_1 - c_2$, a constant. It does not have to equal 0, so the equation from before makes sense. It just tells us that the two constants $c_1, c_2$ differ by 1.

I.9.15: We wish to find $\int \frac{e^x}{\sqrt{1 + e^{2x}}} \, dx$.

First, let $u = e^x$. Then $x = \ln(u)$, so $dx = \frac{1}{u} \, du$. The integral therefore becomes:

$$\int \frac{u}{\sqrt{1 + u^2}} \, \frac{1}{u} \, du = \int \frac{1}{\sqrt{1 + u^2}} \, du$$

We now have to do another substitution to get rid of the square root. Let $u = \tan(\theta)$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $du = \sec^2(\theta) \, d\theta$. Because $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\sec(\theta) > 0$, so $\sqrt{1 + u^2} = \sqrt{1 + \tan^2(\theta)} = \sec(\theta)$. The integral becomes:

$$\int \frac{1}{\sqrt{1 + u^2}} \, du = \int \frac{\sec^2(\theta)}{\sec(\theta)} \, d\theta$$

$$= \int \sec(\theta) \, d\theta$$

$$= \ln |\sec(\theta) + \tan(\theta)| + C$$
We know that \( \tan(\theta) = u = e^x \), and \( \sec(\theta) = \sqrt{1+u^2} = \sqrt{1+e^{2x}} \), so the final answer is:

\[
\ln |\sqrt{1+e^{2x}} + e^x| + C
\]

I.13.3: We want to find \( \int \sqrt{1+x^2} \, dx \). To get rid of the square root, we use a rational substitution. Recall \( U(t) = \frac{1}{2}(t + \frac{1}{t}) \), \( V(t) = \frac{1}{2}(t - \frac{1}{t}) \). Also recall \( U^2 = V^2 + 1 \). We do the substitution \( x = V, t \geq 1 \). So, \( dx = V' \, dt \), and our integral becomes:

\[
\int \sqrt{1+x^2} \, dx = \int \sqrt{1+V^2}V' \, dt = \int UV' \, dt
\]

\[
= \int \frac{1}{2} (t + \frac{1}{t}) \frac{1}{2} (1 + \frac{1}{t^2}) \, dt
\]

\[
= \frac{1}{4} \int t + \frac{2}{t} + \frac{1}{t^2} \, dt
\]

\[
= \frac{1}{4} \left( \frac{t^2}{2} + 2 \ln |t| - \frac{1}{2t^2} \right) + C
\]

\[
= \frac{1}{8} t^2 - \frac{1}{8} t^{-2} + \frac{1}{2} \ln |t| + C
\]

The easiest way to substitute back in the \( x \) is to use the facts (given in the book):

1. \( t = U + V \)
2. \( \frac{1}{t} = U - V \)

So, our expression from before becomes:

\[
\frac{1}{8} (U + V)^2 - \frac{1}{8} (U - V)^2 + \frac{1}{2} \ln |U + V| + C
\]

\[
= \frac{1}{8} (U^2 + 2UV + V^2) - \frac{1}{8} (U^2 - 2UV + V^2) + \frac{1}{2} \ln |U + V| + C
\]

\[
= \frac{1}{2} UV + \frac{1}{2} \ln |U + V| + C
\]

Finally, we can use the fact that \( V = x \) and \( U = \sqrt{1-V^2} = \sqrt{1-x^2} \). So the final answer is:

\[
\frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2} \ln |x + \sqrt{1-x^2}| + C
\]

Alternatively, you could have used the substitution \( x = \tan(\theta) \), and then figured out how to integrate \( \sec^3(\theta) \).

I.13.4: We want to integrate \( \frac{1}{\sqrt{2x-x^2}} \). We first complete the square.

We get \( 2x - x^2 = -(x^2 - 2x) = -(x-1)^2 - 1 = 1 - (x-1)^2 \). Our integral becomes \( \int \frac{dx}{\sqrt{1-(x-1)^2}} \).

This looks like the derivative of \( \arcsin(x-1) \). If you do the substitution, you find that this integrates to \( \arcsin(x-1) + C \).
I.13.7: We want to integrate \( \frac{1}{\sqrt{4-x^2}} \). This looks a lot like the derivative of \( \arcsin \). We factor out a 4 from the square root to get:

\[
\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{2\sqrt{1-(x/2)^2}} = \int \frac{du}{\sqrt{1-u^2}}
\]

This last step was done using the substitution \( u = \frac{x}{2} \). This integrates to \( \arcsin(u) + C \), which equals \( \arcsin\left(\frac{x}{2}\right) + C \).

I.13.15: We want to find \( \int \sqrt{3} \frac{dx}{x^2+1} \). Note that the function within the integral is the derivative of \( \arctan(x) \), so the integral equals \( \arctan(\sqrt{3}) - \arctan(1) \), where we take \( \arctan \) with range in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\).

First, we calculate \( \arctan(\sqrt{3}) \). This equals the angle \( \theta \) in \((-\frac{\pi}{2}, \frac{\pi}{2})\) such that \( \tan(\theta) = \sqrt{3} \). By basic trigonometry, we know this is when \( \theta = \frac{\pi}{3} \), so \( \arctan(\sqrt{3}) = \frac{\pi}{3} \). Next, we find \( \arctan(1) \). This equals the angle \( \theta \) such that \( \tan(\theta) = 1 \), so this equals \( \frac{\pi}{4} \).

The final answer is \( \frac{\pi}{3} - \frac{\pi}{4} \).

I.15.3: Let \( u = x^2 - 1 \). Then \( du = 2xdx \), so the integral becomes:

\[
\int \frac{x}{\sqrt{x^2-1}} dx = \int \frac{1}{2} \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2 - 1} + C
\]

I.15.8: In this problem, we have a ratio of polynomials, so we use polynomial long division and partial fractions. First, we use polynomial long division since the numerator does not have smaller degree, getting:

\[
x^2 - 36)
\begin{array}{c}
\hline
x^4 + 36x^2 \\
-x^4 + 36x^2 \\
\hline
36x^2 \\
-36x^2 + 1296 \\
\hline
1296
\end{array}
\]

So, \( \frac{x^4}{x^2 - 36} = x^2 + 36 + \frac{1296}{x^2 - 36} \). So we get:

\[
\int \frac{x^4}{x^2 - 36} dx = \int x^2 + 36 + \frac{1296}{x^2 - 36} dx = \frac{x^3}{3} + 36x + 1296 \int \frac{1}{x^2 - 36} dx
\]
We do partial fractions on \( \frac{1}{x^2 - 36} = \frac{1}{(x-6)(x+6)} \). This must equal \( \frac{A}{x-6} + \frac{B}{x+6} \) for some \( A, B \). The heavyside method say \( A = \left. \frac{1}{x+6} \right|_{x=6} = \frac{1}{12} \), and \( B = \left. \frac{1}{x-6} \right|_{x=-6} = \frac{-1}{12} \).

So, the fraction equals \( \frac{1}{12(x-6)} - \frac{1}{12(x+6)} \). Integrating this, we get \( \frac{1}{12} \ln |x - 6| - \frac{1}{12} \ln |x + 6| + C \).

The final answer is \( \frac{x^3}{3} + 36x + 1296 \left( \frac{1}{12} \ln |x - 6| - \frac{1}{12} \ln |x + 6| \right) + C \).

**I.15.34:** We have to use partial fractions in both cases.

For the first, we know:

\[
\frac{1}{x(x-1)(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3}
\]

We can use the heavyside method to calculate the coefficients. Using it, we get:

- \( A = \left. \frac{1}{(0-1)(0-2)(0-3)} \right|_{x=0} = -\frac{1}{6} \)
- \( B = \left. \frac{1}{(1)(1-2)(1-3)} \right|_{x=1} = \frac{1}{2} \)
- \( C = \left. \frac{1}{(2)(2-1)(2-3)} \right|_{x=2} = -\frac{1}{2} \)
- \( D = \left. \frac{1}{(3)(3-1)(3-2)} \right|_{x=3} = \frac{1}{6} \)

Integrating, we get \( -\frac{1}{6} \ln |x| + \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x-2| + \frac{1}{6} \ln |x-3| + E \).

For the second, we do almost exactly the same thing. The only thing that is potentially different are the coefficients. Using the heavyside method, they are calculated as follows:

- \( A = \left. \frac{(0)^3 + 1}{(0-1)(0-2)(0-3)} \right|_{x=0} = -\frac{1}{6} \)
- \( B = \left. \frac{(1)^3 + 1}{(1)(1-2)(1-3)} \right|_{x=1} = 1 \)
- \( C = \left. \frac{(2)^3 + 1}{(2)(2-1)(2-3)} \right|_{x=2} = -\frac{9}{2} \)
- \( D = \left. \frac{(3)^3 + 1}{(3)(3-1)(3-2)} \right|_{x=3} = \frac{28}{6} = \frac{14}{3} \)

Integrating, we get \( -\frac{1}{6} \ln |x| + \ln |x-1| - \frac{9}{2} \ln |x-2| + \frac{14}{3} \ln |x-3| + E \).

**I.15.35:** This is a ratio of polynomials, so we’d like to use partial fractions. To do that, we need to factor the denominator, \( 1 + x + x^2 + x^3 \). As the book hints, this equals \( 1 + x + x^2(1 + x) \). Factoring out a \( (1 + x) \), this equals \( (1 + x)(1 + x^2) \). Note that \( 1 + x^2 \) cannot be broken up any more, as we would have to use \( \sqrt{-1} \). So, the two factors are \( 1 + x, 1 + x^2 \).
Based on the method of partial fractions, we find:

\[
\frac{1}{1 + x + x^2 + x^3} = \frac{1}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}
\]

We can’t use the heavyside method because there is a quadratic factor. So, clearing the denominators instead, we get:

\[
1 = A(x^2 + 1) + (Bx + C)(x + 1)
= Ax^2 + A + Bx^2 + Bx + Cx + C
= x^2(A) + x(B + C) + A + C
\]

We must have \( A = 0, B + C = 0, A + C = 1 \). So, \( A = 0, B = -1, C = 1 \). Therefore, our integral is:

\[
\int \frac{dx}{1 + x + x^2 + x^3} = \int \frac{-x + 1}{x^2 + 1} dx
= -\int \frac{x}{x^2 + 1} dx + \frac{1}{x^2 + 1}
\]

To solve the first integral, we use \( u = x^2 + 1 \), so \( du = 2x dx \). The integral becomes:

\[
-\int \frac{\frac{1}{2}du}{u} = -\frac{1}{2} \ln |u| = -\frac{1}{2} \ln |x^2 + 1|
\]

The second integral is simply \( \arctan(x) \), so the final answer is \( -\frac{1}{2} \ln |x^2 + 1| + \arctan(x) + E \).

**I.15.38:** Once you’ve drawn the graph, note that it is symmetric around the \( x \)-axis. That is, you can reflect it to get the same graph. What this means is that any area above the \( x \)-axis (positive area) cancels out with the area below the \( x \)-axis, so the area under the curve is 0.

**I.15.43:** While the book claims that the function \( F(x) \) is an anti-derivative for \( f(x) \), it is not! An antiderivative needs to have the correct derivative on the entire interval \([0, 2]\). But \( F(x) \) is not differentiable at the point \( x = 1 \). In fact, \( F(x) \) is not even continuous here!. You get different values whether you are approaching 1 from the left or the right.

We have:

\[
\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} \left( \frac{1}{2}x^2 - x \right) = -\frac{1}{2}
\]

\[
\lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} \left( x - \frac{1}{2}x^2 \right) = \frac{1}{2}
\]

Since \( F(x) \) is not continuous at 1, it is not differentiable for 1, and so is not an antiderivative for \( f(x) \) on the interval \([0, 2]\).

**I.15.44:** The issue here comes from the substitution. Remember that \( x \) has domain \([-1, 1]\), and \( u = 1 - x^2 \). Therefore, the book claims that \( x = \sqrt{1 - u} \). However, \( \sqrt{1 - u} \geq 0 \) for all \( u \), but \( x \) needs to be negative at times. This substitution does not work because it does not pass the horizontal line test on the interval \([-1, 1]\). When you graph \( u = 1 - x^2 \) on this interval, you will see that it fails this test.