Math 763. Homework 7

Due Thursday, November 14th

A morphism $f : X \to Y$ is a *fibration* (or a fiber bundle, or a locally trivial family) with fiber Z if each point $y \in Y$ has a neighborhood $V \ni y$ such that the preimage $f^{-1}(V)$ is isomorphic to $V \times Z$; moreover, the isomorphism must transform the restriction $f : f^{-1}(V) \to V$ into the projection $V \times Z \to V$.

1. Let $Gr = Gr(k+1, n+1)$ be the Grassmannian of k-dimensional projective subspaces in $\mathbb{P}^n$. Let $X \subset Gr \times \mathbb{P}^n$ be the incidence relation. Show that the projections $X \to Gr$ and $X \to \mathbb{P}^n$ are fibrations.

2. Let $Mat(n, m)$ be the space of $n \times m$ matrices, considered as the affine space of dimension nm. Fix r, and let $X \subset Mat(n, m)$ be the set of matrices of rank exactly r. Consider X as a (quasi-affine) algebraic variety. For any $A \in X$, the kernel $\ker(A)$ is a subspace of k^m of dimension $m - r$, while the image $\mathrm{im}(A)$ is a subspace of k^n of dimension r. Prove that the maps

$$\ker : X \to Gr(m-r, m) : A \mapsto \ker(A)$$
$$\mathrm{im} : X \to Gr(r, n) : A \mapsto \mathrm{im}(A)$$

are morphisms of varieties.

3. Keeping the notation of the previous problem, show that the map

$$(\ker, \mathrm{im}) : X \to Gr(m-r, m) \times Gr(r, n)$$

is a fibration. (This implies that the maps ker and im are fibrations as well.)

4. Let Y be any variety, and suppose $X \subset Y \times \mathbb{P}^n$ is a closed subset. Fix $d < n$, and let Z be the locus of $y \in Y$ such that the fiber $X \cap \{y\} \times \mathbb{P}^n$ contains a d-dimensional projective subspace (the fiber is a closed subset of $\mathbb{P}^n$). Prove that $Z \subset Y$ is closed. Note: here 'projective subspace' means a 'linearly embedded projective space of smaller dimension', for instance, a line if $d = 1$, a plane if $d = 2$, etc.

5. Let $f_0, \ldots, f_n$ be $n+1$ homogeneous polynomials of fixed degrees $d_0, \ldots, d_n > 0$ in $n+1$ variables $x_0, \ldots, x_n$. Prove that there exists an expression D, polynomial in the coefficients of f_i's, such that $D = 0$ if and only if the system of equations $f_0 = \cdots = f_n = 0$ has non-trivial solutions. (One classical special case of this is $d_0 = \cdots = d_n = 1$; the other is $n = 1$.)

6. (déjà vu) Prove that a generic degree d hypersurface in $\mathbb{P}^n$ contains no lines if $d > 2n - 3$ (and $n > 1$). More precisely, let V_d be the space of degree d homogeneous polynomials in $n+1$ variables. Prove that there exists there exists a non-empty Zariski open subset $U \subset V_d$ such that for any $f \in U$, the hypersurface $f = 0$ contains no lines.