Math 763. Homework 8

Due Thursday, November 21st

Let us call a morphism $f : X \to Y$ *projective* if there exists n and a map $g : X \to \mathbb{P}^n$ such that the morphism $(g, f) : X \to \mathbb{P}^n \times Y$ is a closed embedding. For example, if Y is a point, $f : X \to Y$ is projective if and only if X is a projective variety.

(Remark: This is not the only possible definition of a projective morphism: there are several slightly different notions, and it is not always clear which one works better. This is perhaps the most restrictive version.)

The importance of this class of morphisms is that the two theorems proved in class (closedness of image and semi-continuity of fibers) hold for projective morphisms, more or less tautologically (do you see why?)

Let us verify that this class of morphisms behaves 'nicely':

1. (Compatibility with composition) Show that if $f : X \to Y$ and $g : Y \to Z$ are projective, then so is $f : X \to Z$.

2. (Compatibility with base change = fiber product) Show that if $f : X \to Y$ is projective and $g : Z \to Y$ is arbitrary, then the natural morphism $X \times_Y Z \to Z$ is projective.

3. (Compatibility with product) If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are projective, then so is $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$. (By the way, this formally follows from the two previous problems.)

4. Any morphism $f : X \to Y$ with projective X and separated Y is projective.

5. A closed embedding $f : X \to Y$ is projective.

6. Let $f : X \to Y$ be a finite map of affine varieties: that is, X is affine, Y is affine, and $k[X]$ is a finitely-generated $k[Y]$-module. Prove that f is projective.

And here is one more problem to practice with Grassmannians:

7. Let X and Y be closed subvarieties in the same projective space $\mathbb{P}^n$. Suppose $X \cap Y \neq \emptyset$. For every $x \in X$ and $y \in Y$, let $\ell_{x,y}$ be the line passing through x and y. Show that the subset

$$\bigcup_{x \in X, y \in Y} \ell_{x,y} \subset \mathbb{P}^n$$

is closed. (A special case of this construction is when Y is a point, in which case the result looks like a "projective cone".)