Math 764. Homework 1

Due Friday, February 3rd

In all these problems, we fix a topological space X; all sheaves and presheaves are sheaves on X.

Example:

1. Let X be the unit circle, and let $\mathcal{F}$ be the sheaf of C^∞-functions on X. Find the (sheaf) image and the kernel of the morphism

$$\frac{d}{dt} : \mathcal{F} \to \mathcal{F}.$$

Here $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the polar coordinate on the circle.

Operations on sheaves:

2. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of sets. Recall that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a (categorical) monomorphism if and only if for any sheaf $\mathcal{F}'$ and any two morphisms $\psi_1, \psi_2 : \mathcal{F}' \to \mathcal{F}$, the equality $\phi \circ \psi_1 = \phi \circ \psi_2$ implies $\psi_1 = \psi_2$. Show that ϕ is a monomorphism if and only if it induces injective maps on all stalks.

3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of sets. Recall that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a (categorical) epimorphism if and only if for any sheaf $\mathcal{G}'$ and any two morphisms $\psi_1, \psi_2 : \mathcal{G} \to \mathcal{G}'$, the equality $\psi_1 \circ \phi = \psi_2 \circ \phi$ implies $\psi_1 = \psi_2$. Show that ϕ is a epimorphism if and only if it induces surjective maps on all stalks.

4. Show that any morphism of sheaves can be written as a composition of an epimorphism and a monomorphism. (You should know what order of composition I mean here.)

5. Let $\mathcal{F}$ be a sheaf, and let $\mathcal{G} \subset \mathcal{F}$ be a sub-presheaf of $\mathcal{F}$ (thus, for every open set $U \subset X$, $\mathcal{G}(U)$ is a subset of $\mathcal{F}(U)$ and the restriction maps for $\mathcal{F}$ and $\mathcal{G}$ agree). Show that the sheafification $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is naturally identified with a subsheaf of $\mathcal{F}$.

6. Let $\mathcal{F}_i$ be a family of sheaves of abelian groups on X indexed by a set I (not necessarily finite). Show that the direct sum and direct product of this family exists in the category of sheaves of abelian groups. (E.g., a direct sum would be a sheaf of abelian groups $\mathcal{F}$ together with a universal family of homomorphisms $\mathcal{F}_i \to \mathcal{F}$.) Do these operations agree with (a) taking stalks at a point $x \in X$ (b) taking sections over an open subset $U \subset X$?

Locally constant sheaves:

Definition. A sheaf $\mathcal{F}$ is *constant over an open set* $U \subset X$ if there is a subset $S \subset F(U)$ such that the map

$$\mathcal{F}(U) \to \mathcal{F}_x : s \mapsto s_x \text{ (the germ of } s \text{ at } x)$$

gives a bijection between S and $\mathcal{F}_x$ for all $x \in U$.

$\mathcal{F}$ is *locally constant* (on X) if every point of X has a neighborhood on which $\mathcal{F}$ is constant.

7. Recall that a *covering space* $\pi : Y \to X$ is a continuous map of topological spaces such that every $x \in X$ has a neighborhood $U \ni x$ whose preimage $\pi^{-1}(U) \subset U$ is homeomorphic to $U \times Z$ for some discrete topological space Z. (Z may depend on x; also, the homeomorphism is required to respect the projection to U.)

Show that if $\pi : Y \to X$ is a covering space, its sheaf of sections $\mathcal{F}$ is locally constant. Moreover, prove that this correspondence is an equivalence between the category of covering spaces and the category of locally constant sheaves. (If X is pathwise connected, both categories are equivalent to the category of sets with an action of the fundamental group of X.)

Sheafification:

8. (This problem may be hard, but it is still a good idea to try it) Prove or disprove the following statement (contained in the lecture notes). Let $\mathcal{F}$ be a presheaf on X, and let $\tilde{\mathcal{F}}$ be its sheafification. Then every section $s \in \tilde{\mathcal{F}}(U)$ can be represented as (the equivalence class of) the following gluing data: an open cover $U = \bigcup U_i$ and a family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.