Math 764. Homework 2  
Due Friday, February 10th

Extension of a sheaf by zero.
Let $X$ be a topological space, let $U \subset X$ be an open subset, and let $\mathcal{F}$ be a sheaf of abelian groups on $U$.

1. The extension by zero $j_!\mathcal{F}$ of $\mathcal{F}$ (here $j$ is the embedding $U \hookrightarrow X$) is the sheaf on $X$ that can be defined as the sheafification of the presheaf $\mathcal{G}$ such that
\[
\mathcal{G}(V) = \begin{cases} 
\mathcal{F}(V), & V \subset U \\
0, & V \nsubseteq U.
\end{cases}
\]

Is the sheafication necessary in this definition? (Or maybe $\mathcal{G}$ is a sheaf automatically?)

2. Describe the stalks of $j_!\mathcal{F}$ over all points of $X$ and the espace étalé of $j_!\mathcal{F}$.

3. Verify that $j_!$ is the left adjoint of the restriction functor from $X$ to $U$: that is, for any sheaf $\mathcal{G}$ on $X$, there exists a natural isomorphism
\[
\text{Hom}(\mathcal{F}, \mathcal{G}|_U) \simeq \text{Hom}(j_!\mathcal{F}, \mathcal{G}).
\]
(The restriction $\mathcal{G}|_U$ of a sheaf $\mathcal{G}$ from $X$ to an open set $U$ is defined by $\mathcal{G}|_U(V) = \mathcal{G}(V)$ for $V \subset U$.)

Side question (not part of the homework): What changes if we consider the version of extension by zero for sheaves of sets (‘the extension by empty set’)?

Examples of affine schemes.

4. Let $R_\alpha$ be a finite collection of rings. Put $R = \prod_\alpha R_\alpha$. Describe the topological space $\text{Spec}(R)$ in terms of $\text{Spec}(R_\alpha)$’s. What changes if the collection is infinite?

5. Recall that the image of a regular map of varieties is constructible (Chevalley’s Theorem); that is, it is a union of locally closed sets. Give an example of a map of rings $R \to S$ such that the image of a map $\text{Spec}(S) \to \text{Spec}(R)$ is

(a) An infinite intersection of open sets, but not constructible.
(b) An infinite union of closed sets, but not constructible. (This part may be very hard.)

Contraction of a subvariety.
Let $X$ be a variety (over an algebraically closed field $k$) and let $Y \subset X$ be a closed subvariety. Our goal is to construct a $k$-ringed space $Z = (Z, \mathcal{O}_Z) = X/Y$ that is in some sense the result of ‘gluing’ together the points of $Y$. While $Z$ can be described by a universal property, we prefer an explicit construction:

- The topological space $Z$ is the ‘quotient-space’ $X/Y$: as a set, $Z = (X - Y) \sqcup \{z\}$; a subset $U \subset Z$ is open if and only if $\pi^{-1}(U) \subset X$ is open. Here the natural projection $\pi : X \to Z$ is identity on $X - Y$ and sends all of $Y$ to the ‘center’ $z \in Z$.
- The structure sheaf $\mathcal{O}_Z$ is defined as follows: for any open subset $U \subset Z$, $\mathcal{O}_Z(U)$ is the algebra of functions $g : U \to k$ such that the composition $g \circ \pi$ is a regular function $\pi^{-1}(U) \to k$ that is constant along $Y$. (The last condition is imposed only if $z \in U$, in which case $Y \subset \pi^{-1}(U)$.)
In each of the following examples, determine whether the quotient $X/Y$ is an algebraic variety; if it is, describe it explicitly.

6. $X = \mathbb{P}^2, Y = \mathbb{P}^1$ (embedded as a line in $X$).

7. $X = \{(s_0, s_1; t_0 : t_1) \in \mathbb{A}^2 \times \mathbb{P}^1 : s_0 t_1 = s_1 t_0\}, Y = \{(s_0, s_1; t_0 : t_1) \in X : s_0 = s_1 = 0\}$.

8. $X = \mathbb{A}^2, Y$ is a two-point set (if you want a more challenging version, let $Y \subset \mathbb{A}^2$ be any finite set).