Math 764. Homework 3
Due Friday, February 17

1. (Gluing morphisms of sheaves) Let F and G be two sheaves on the same space X. For any open set $U \subset X$, consider the restriction sheaves $F|_U$ and $G|_U$, and let $Hom(F|_U, G|_U)$ be the set of sheaf morphisms between them.

Prove that the presheaf on X given by the correspondence

$$U \mapsto Hom(F|_U, G|_U)$$

is in fact a sheaf.

2. (Gluing morphisms of ringed spaces) Let X and Y be ringed spaces. Denote by $\underline{Mor}(X, Y)$ the following pre-sheaf on X: its sections over an open subset $U \subset X$ are morphisms of ringed spaces $U \to Y$ where U is considered as a ringed space. (And the notion of restriction is the natural one.) Show that $\underline{Mor}(X, Y)$ is in fact a sheaf.

3. (Affinization of a scheme) Let X be an arbitrary scheme. Prove that there exists an affine scheme X_{aff} and a morphism $X \to X_{aff}$ that is universal in the following sense: any map form X to an affine scheme factors through it.

4. Let us consider direct and inverse limits of affine schemes. For simplicity, we will work with limits indexed by positive integers.

(a) Let R_i be a collection of rings ($i > 0$) together with homomorphisms $R_i \to R_{i+1}$. Consider the direct limit $R := \varinjlim R_i$. Show that in the category of schemes,

$$\operatorname{Spec}(R) = \varprojlim \operatorname{Spec} R_i.$$

(b) Let R_i be a collection of rings ($i > 0$) together with homomorphisms $R_{i+1} \to R_i$. Consider the inverse limit $R := \varprojlim R_i$. Show that generally speaking, in the category of schemes,

$$\operatorname{Spec}(R) \neq \varinjlim \operatorname{Spec} R_i.$$

5. Here is an example of the situation from 4(b). Let k be a field, and let $R_i = k[t]/(t^i)$, so that $\varprojlim R_i = k[[t]]$. Describe the direct limit

$$\varinjlim \operatorname{Spec} R_i$$

in the category of ringed spaces. Is the direct limit a scheme?

6. Let S be a finite partially ordered set. Consider the following topology on S: a subset $U \subset S$ is open if and only if whenever $x \in U$ and $y > x$, it must be that $y \in U$.

Construct a ring R such that $\operatorname{Spec}(R)$ is homeomorphic to S.

7. Show that any quasi-compact scheme has closed points. (It is not true that any scheme has closed points!)

8. Give an example of a scheme that has no open connected subsets. In particular, such a scheme is not locally connected. Of course, my convention here is that the empty set is not connected...