Math 764. Homework 4
Due Wednesday, March 4th

1. Let X be a variety and let L be a vector bundle on X. For any n, consider the total space of L and the total space of n-th tensor power $L^{\otimes n}$ of L. There is a natural map

$$\nu : L \to L^{\otimes n} : v \mapsto v^{\otimes n}$$

The map is not a morphism of vector bundles, because on fibers, it is not linear, but rather homogeneous of degree n. Prove that the map is regular.

2. (Continuation: n-th root of a section) Now suppose X is smooth and $s : X \to L^{\otimes n}$ is a regular section. The image $s(X) \subset L^{\otimes n}$ is a closed subvariety; therefore, the preimage $\nu^{-1}(s(X)) \subset L$ is a subvariety of the total space of L. Note that this preimage is not a section, but rather multivalued (n-valued) section: its projection to X is generally speaking n-to-1. It is called the n-th root of the section s.

Assume that $s \neq 0$ and that the characteristic of the ground field does not divide n. Show that this subvariety is smooth if and only if the following condition holds: the zero divisor of s is a sum of disjoint smooth prime divisors (all with multiplicity one).

3. (Exact sequence of vector bundles) Let L be the tautological line bundle on $\mathbb{P}^1$. Recall that there are two linearly independent sections s_0, s_1 of L^{-1}. We can use them as components of a morphism of vector bundles

$$\phi : \mathbb{P}^1 \times \mathbb{A}^2 \to L^{-1} : (x, a_0, a_1) \mapsto a_0 s_0(x) + a_1 s_1(x).$$

(The source is the trivial bundle of rank 2).

Prove that this map is surjective. Therefore, its kernel $\ker(\phi)$ is a subbundle of the trivial vector bundle. Find this subbundle (since $\operatorname{Pic} \mathbb{P}^1 \simeq \mathbb{Z}$, this subbundle is isomorphic to L^k for some $k \in \mathbb{Z}$. The problem is to find k.)

This can be viewed as an exact sequence of vector bundles

$$0 \to \ker(\phi) \to \mathbb{P}^1 \times \mathbb{A}^2 \to L^{-1} \to 0.$$

4. Show that the exact sequence from the previous problem does not split; that is, the trivial bundle is not isomorphic to the direct sum of $\ker(\phi)$ and L^{-1}. (This would follow from an appropriately general version of the Krull-Schmidt Theorem, but it may be easier to prove this directly.)

5. Let X be a projective smooth irreducible variety. The *canonical class* $K_X \in \mathrm{Pic}(X)$ of X is the image of the *canonical bundle* $\bigwedge^{dim(X)} T^*X$ in $\mathrm{Pic}(X)$. Compute the canonical class of $\mathbb{P}^n$. (Recall that $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.)

6. (The Euler exact sequence). Let L be the tautological line bundle on $\mathbb{P}^n$. Recall there are $n+1$ linearly independent sections $s_0, \ldots, s_n \in \Gamma(\mathbb{P}^n, L^{-1})$; we may consider them as components of a single section $s = (s_0, \ldots, s_n) : \Gamma(\mathbb{P}^n, V)$, where the vector bundle V is the direct sum of $n+1$ copies of L^{-1}.

The section s may be viewed as a morphism from the trivial vector bundle $\mathbb{P}^n \times \mathbb{A}^1$ to V. Prove that the morphism is injective, and that its cokernel (that is, the quotient of the target by its image) is isomorphic to $T\mathbb{P}^n$.