Math 222 HW 7 solutions.

Note Title 3/12/2014

III. 4

1. $\frac{dy}{dx} = xy \quad \int \frac{1}{y}\,dy = \int x\,dx \quad \ln|y| = \frac{1}{2}x^2 + C$

$|y| = e^C \cdot e^{\frac{1}{2}x^2}$. Since $e^C \cdot e^{\frac{1}{2}x^2} > 0$

and $y(2) = -1 < 0$, $y = -e^C e^{\frac{1}{2}x^2}$

$-1 = -e^C \cdot e^2 \quad C = -2.$

2. $\frac{dy}{dx} = -x\cos^2 y. \quad \int \frac{dy}{\cos^2 y} = \int -x\,dx$

$\tan y = -\frac{1}{2}x^2 + C.$

$y = \arctan(-\frac{1}{2}x^2 + C)$

$\frac{\pi}{3} = \arctan(c)$ So, $C = \sqrt{3}.$

3. $\frac{dy}{dx} = -\frac{1+x}{1+y}. \quad \int (1+x)\,dx = \int -(1+y)\,dy$

$\frac{1}{2}x^2 + x = -y - \frac{1}{2}y^2 + C.$

i.e $x^2 + 2x + y^2 + 2y = C.$

Since $y(0) = A$, $C = A^2 + 2A.$

$x^2+2x+y^2+2y = A^2+2A$

4. $y^2\frac{dy}{dx}+x^3=0$, $y(0)=A$.

$\int y^2 dy = \int -x^3 dx \qquad \frac{1}{3}y^3 = -\frac{1}{4}x^4+C$.

$y = \left(-\frac{3}{4}x^4+C\right)^{1/3}$.

$y(0)=A=(C)^{1/3}$, So, $C=A^3$.

5. $\frac{dy}{dx}=y^2-1$. $\int \frac{dy}{y^2-1} = \int dx$

$\int \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\,dy = \int dx$

$\frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = x+c$

$\ln\left|\frac{y-1}{y+1}\right| = 2x+C. \qquad \left|\frac{y-1}{y+1}\right| = e^{2x}\cdot e^{c}$

So $\frac{y-1}{y+1} = \pm e^{c}\cdot e^{2x}$.

If $\frac{y(0)-1}{y(0)+1} = \frac{A-1}{A+1} \geq 0$, then

$\frac{y-1}{y+1} = e^{c}\cdot e^{2x}$, and $c=\ln\frac{A-1}{A+1}$.

Otherwise, $\frac{y-1}{y+1} = -e^c \cdot e^{2x}$ and $c = \ln\left(-\frac{A-1}{A+1}\right)$.

In either case, solve y in x.

6. $\frac{dy}{dx} = -y^2-1$ $\quad \int \frac{dy}{y^2+1} = \int -1\, dx$

$c - x = \arctan y$. So $y = \tan(c-x)$.

$A = \tan(c)$. $\quad c = \arctan A$.

7. $\frac{dy}{dx} = \frac{1-x^2}{y}$. $\quad \int y\, dy = \int 1-x^2 dx$

$\frac{1}{2}y^2 = x - \frac{1}{3}x^3 + C$. $\quad y(0) = 1$

gives $C = \frac{1}{2}$.

8. Notice: for $p < 1000$, $\frac{dp}{dt} > 0$.

and for $P > 1000$, $\frac{dP}{dt} < 0$.

So, $100 \leq p < 1000$

(a) $\int \frac{50}{p(1000-p)} dp = \int dt$

$$\int \frac{\frac{1}{20}}{p} + \frac{\frac{1}{20}}{1000-p}\, dp = t + c.$$

$\ln p - \ln(1000-p) = t + c$, since $1000-p > 0$.

$$\ln \frac{p}{1000-p} = t + c$$

$$\frac{p}{1000-p} = e^t \cdot e^c \qquad p = \frac{1000e^c \cdot e^t}{1 + e^c \cdot e^t}$$

(b) When $t=0$, $p=100$, so $e^c = 1/9$.

$$p = \frac{1000\, e^t}{9 + e^t}$$

(c) Solving $500 = \dfrac{1000e^t}{9+e^t}$

$$4500 + 500e^t = 1000e^t$$

$$e^t = 9 \qquad t = 2\ln 3$$

(d) Solve $900 = \dfrac{1000e^t}{9+e^t}$

$$8100 + 900e^t = 1000e^t$$

$$8100 = 100e^t \qquad e^t = 81 \qquad t = 4\ln 3$$

(e). $\frac{dp}{dt} = \frac{1}{50} p(1000-p)$ is a.

probala. the max occurs at the mid-point of the two zeros.

So, $\frac{dp}{dt}$ is maximized, when $p=500$.

II. 6.

1. The purpose of multiplying both sides of certain $m(x)$ is that $m(x)$ is chosen in such a way that the left side of the equation will be an application of the product rule, i.e. $\frac{d}{dx}(m \cdot y) = m \cdot \frac{dy}{dx} + \frac{dm}{dx} \cdot y$

Taking $m(x) = 0$ would give the equation $0 = 0$, which doesn't help solve the original equation at all.

2. The $m(x)$ inside the integral is part of the integrand.
Such cancellation is incorrect unless $m(x)$ is a constant.

3. $$\frac{dy}{dx} + y = x$$

$$m(x) = e^{\int 1\,dx} = e^x.$$

$$m(x)\frac{dy}{dx} + m(x)y = xe^x$$

$$\frac{d}{dx}(m(x)\cdot y) = xe^x$$

$$m(x)\cdot y = \int xe^x dx = xe^x - e^x + C.$$

$$y = C\cdot e^{-x} + x - 1.$$

Since $y(0) = 0$, $C = 1$.

4. $$\frac{dy}{dx} - 2y = x^2.$$

$$m(x) = e^{\int -2dx} = e^{-2x}$$

$$\frac{d}{dx}(m(x)\cdot y) = m(x)\,x^2.$$

$$m(x)\cdot y = \int x^2\cdot e^{-2x}\,dx$$

$$= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx$$

$$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \int e^{-2x}\,dx$$

$$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{2}e^{x} + c$$

Since $y(0)=0$, $c = \frac{1}{2}$