Matrices

Determinant:

- $\det(AB) = \det(A)\det(B)$.
- If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then $\det(A) = \lambda_1 \cdots \lambda_n$.

Trace:

- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and hence $\mathrm{tr}(AB - BA) = 0$.

Characteristic polynomial:

- Given a $n \times n$ matrix A, the characteristic polynomial $p_A(t) = \det(tI - A)$, where I is the identity matrix.
- The Cayley–Hamilton theorem states that $p_A(A) = 0$.

Jordan form and diagonalize matrices:

- An $n \times n$ matrix A is diagonalizable, i.e., there exists invertible matrix P with $P^{-1}AP$ diagonal matrix, if and only if the sum of the dimensions of the eigenspaces is n.
- Under a base change, every $n \times n$ matrix A can be written in the Jordan normal form. The sizes of the Jordan blocks do not depend on the choice of the base change.
- A set of square matrices is said to be simultaneously diagonalizable if there exists a single invertible matrix P such that $P^{-1}AP$ is a diagonal matrix for every A in the set. A set of diagonalizable matrices commutes if and only if the set is simultaneously diagonalizable.

1. (Vandermonde determinant) Let $x_1, x_2, \dots, x_n$ be arbitrary numbers. Compute the determinant
$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_{n-1}^{n-1} \end{vmatrix}.$$

2. Let F_n be the Fibonacci sequence ($F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$). Prove that
$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n, \text{ for } m, n \geq 0.$$

3. Let A and B be $n \times n$ matrices with real entries satisfying

$$\operatorname{tr}(AA^t + BB^t) = \operatorname{tr}(AB + A^tB^t).$$

Prove that $A = B^t$.

4. Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

5. Let X, Y be $n \times n$ matrices. Prove that

$$\det(I_n - XY) = \det(I_n - YX),$$

where I_n is the $n \times n$ identity matrix.

6. Let $A = (a_{ij})_{ij}$ be an $n \times n$ matrix such that $\sum_{j=1}^{n} |a_{ij}| < 1$ for each i. Prove that $I_n - A$ is invertible.

7. Given two $n \times n$ matrices A and B for which there exists nonzero real numbers a and b such that

$$AB = aA + bB.$$

Prove that A and B commute.

8. Let M be an invertible $2n \times 2n$ matrix, represented in block form as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } M^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$

Show that $\det M \cdot \det H = \det A$.

9. Let A be the $n \times n$ matrix whose (i, j)-entry is equal to $\frac{1}{\min(i,j)}$ for $1 \leq i, j \leq n$. Compute $\det(A)$.

10. Do there exist polynomials $a(x)$, $b(x)$, $c(x)$, $d(x)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?