11/16/11 – Mock Putnam

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that there is some real number a such that $f(f(f(a))) = a$. Show that there is some real number b such that $f(b) = b$.

2. Let $g(n)$ be the number of ways to write n as the ordered sum of positive integers, at least one of which is even and at least one of which is odd. Find, with proof, $g(11)$ and $g(12)$.

3. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1 x + 1$ be a polynomial, where the a_i $(1 \le i \le 97)$ are real numbers. Prove that $P(x) = 0$ has at least one complex root (i.e., a root of the form $a + bi$ with a, b real numbers and $b \neq 0$). (VTRMC 2011)

4. Find, with proof, the number of ordered pairs of integers (m, n) such that $\dfrac{1}{m} + \dfrac{1}{n} = \dfrac{1}{91}$.

5. Find, with proof, all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative such that $xf(x) = f(x^2)$ holds for all $x \in \mathbb{R}$.