# Ordered Sets

Putnam Club 2019

## Basics

**Cartesian Product**: Let $A$ and $B$ be sets. The cartesian product $A \times B$ is the set of ordered pairs you can create from $A$ and $B$:

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

**Relation**: A relation $R$ from $A$ to $B$ is any subset of $A \times B$. If $A = B$ then we say $R$ is a relation on $A$.
**Note:** A relation is a mathy way of describing a "yes/no" question between two sets. For example a relation on $\mathbb{N}$ can be given by "divides". That is:

$$R = \left\{(n, m) \in \mathbb{N}^2 : n|m\right\}.$$

In this case if the answer to the question "Does $n$ divide $m$?" is "Yes." then $(n, m)$ is an element of $R$.

Note that in general we use lots of symbols in lieu of $R$. In the above case we would just use the symbol $|$ for "divides" and not refer to $R$ at all.
**Exercise** Explain how the following can be thought of as relations: functions, equality, less-than.
**Ordering** An order on set $A$ is a relation $R$ on $A$ which has the following three properties:

1. For all $a \in A$ we have $(a, a) \in R$.
2. If $(a, b)$ and $(b, a)$ are in $R$ then $a = b$.
3. If $(a, b)$ and $(b, c)$ are in $R$ then $(a, c)$ is also.

In general the symbol we use for an ordering is $\leq$ but do not think if you see this symbol we are only looking at the standard notion of "less than or equal to" on the real numbers!
**Exercises**

1. Explain that equality is an ordering.
2. Let $A$ be the set of all subway stations in some city. Consider the relation on $A$ given by $a \leq b$ whenever you can travel from station $a$ to $b$ by trains and $b$ is not west of $a$. Is this an ordering? If it is not an ordering, can you add conditions which would make it one?

**Strict Ordering** A strict ordering on a set $A$ replaces the first two properties of an ordering with:

1. For all $a \in A$ we have $(a, a)$ not in $R$.
2. If $(a, b) \in R$ then $(b, a)$ is not in $R$.

**Exercise**

1. Let $A$ be the set of all possible placements of any number of any types of chesspieces on a chessboard. Let $(a, b) \in R$ if you can move from placement $a$ to $b$ by a finite sequence of allowable moves. Is this a strict ordering?
2. Consider the game tic-tac-toe where X is the first to play. Let $A$ be the set of all possible legal arangments of Xs and Os. Can you construct a strict ordering on $A$?
3. Consider the relation on $\mathbb{C}$ where $(w, z) \in R$ when $|w| < |z|$. Classify this as an ordering, a strict ordering, or neither.

**Definition** A total ordering is an ordering for which all elements are comparable. That is for all $a, b \in A$ either $(a, b)$ or $(b, a)$ is an element of the relation.
**Exercise**

1. Decide if the relations given in the previous exercise are necessarily total orderings.

2. Construct a total ordering on $\mathbb{R}^2$ or explain why it is not possible.

**Maximal Elements** Let $A$ be an ordered set. A maximal element $M$ of $A$ is an element with the property that if $M \leq a$ then $M = a$.
**Exercises:**

1. Define a minimal element on an ordered set.

2. Prove that every finite ordered set has a maximal (and a minimal) element. Does it need to be unique?

3. Give an example of an ordered set with no maximal element.

4. Prove that every finite totally ordered set had a unique maximal element.

**Well ordered** A totally ordered set $A$ is called well ordered if every nonempty subset of $A$ has a minimal element.
**Exercise**

1. Show that the positive real numbers are not well ordered under the standard ordering. Moreover, show that you can

2. Prove that $\mathbb{N}$ is well ordered the standard ordering.

# Problem Solving Strategies

The essential strategies:

- Exploit a cannonical ordering (of some sort) on a set.
  - For example, if you are dealing with a finite subest of real numbers you can immediatly order them from smallest to largest. Then you might see how they are distributed and leverage a pigeonhole type principle: "How large does a set of natural numbers less than 100 have to be in order to guarantee at least one even number?" Or "Given 50 even positive integers, why must one of them be larger than 100?"
- Construct your own ordering (of some sort) on a set.
- Identify interesting elements based on an ordering and exploit them (maximums, minimums, infimums and supremums).
  - This often results in useful arguments by contradiciton: You want to argue that a finite totally ordered set has an element with property $p$. Take the minimum from the set and suppose that it does not have property $p$. Then prove that this is a contradiciton, in which case the minimum MUST have the property.

**Examples**

1. Given 50 distinct positive integers less than 100 at least two of them must be coprime.

2. Given a finite colelction of squares whose area sums to 1 show that they can be arranged inside a square with area 2.

3. Given $n$ points in the plane, no three of which are colinear, show that there exists a closed polygon having these points as vertices.

# Exercises

1. Consider a planar region of area one obtained as the union of finitely many disks. Prove that from these disks we can select a subset of mutually disjoint disks with area at least 1/9.

2. Prove that among any eight positive integers less than 2004 we can find numbers $a, b, c, d$ so that

$$4 + d \leq a + b + c \leq 4d.$$

3. Let $\{a_k\}$ be a sequence of distinct positive integers. Show that for any $n$ we have

$$\sum_{k=1}^{n} a_k^2 \geq \frac{2n+1}{3} \sum_{k=1}^{n} a_k.$$

4. Let $X$ be a subset of positive integers closed under addition. Let $Y$ be the complement of $X$. Then

$$\sum_{x \in Y} x \leq |Y|^2.$$

5. Let $P$ be a polynomial with integer coefficients with degree at least two. Show that the set $A = \{x \in \mathbb{Z} : P(P(x)) = x\}$ has at most $n$ elements.

6. You have a party inviting friends from High School and friends from College. Assume that they are distinct sets. At the end of the party you learn:

   - None of your high school friends met all of your college friends.
   - Every one of your college friends met at least one of your high school firends.

   Show that there are two pairs of friends $(a, b)$ and $(c, d)$ who met each other, but $a$ did not meet $d$ and $c$ did not meet $b$.

7. Find all odd positive integers $n$ greater than one such that if $a$ and $b$ are coprime and divide $n$ then $a + b - 1$ is also a divisor of $n$.

8. Consider a matrix in which all entries are real numbers whose absolute value does not exceed one. Moreover, the column sums of the matrix are all zero. Show that you can permute the entries in each column so that the row sums will have absolute value no larger than two.