Wednesday, March 6th and March 13th

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Algebra

Last time, on March 6th, we covered pages 25-39 from *Putnam and Beyond*. Please read the material before attempting to solve the problems. Also, write your solutions as detailed as you can and hand them out on Wednesday, March 13th, in class. You are not required to solve the problems, but they are fun and I promise you will enjoy it! Good luck!

1. Let a, b, c be distinct positive integers and let k be a positive integer such that
$$ab + bc + ca \geq 3k^2 - 1.$$
Prove that
$$\frac{a^3 + b^3 + c^3}{3} - abc \geq 3k.$$

2. Let a, b, c be the side lengths of a triangle. Prove that
$$\left(\frac{a^3 + b^3 + c^3 + 3abc}{2}\right)^{\frac{1}{3}} \geq \max(a, b, c).$$

3. Find all integers that can be represented as $a^3 + b^3 + c^3 - 3abc$ for some positive integers a, b, and c.

4. Find all pairs (x, y) of integers such that
$$xy + \frac{x^3 + y^3}{3} = 2007.$$

5. Let k be an integer and let
$$n = \sqrt[3]{k + \sqrt{k-1}} + \sqrt[3]{k - \sqrt[2]{k^2 - 1}} + 1.$$
Prove that $n^3 - 3n^2$ is an integer.

6. Let a, b, c be positive integers such that $ab + bc + ca \geq 3$. Prove that
$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}.$$

7. Let a, b, c be positive real numbers. Prove that
$$\frac{a}{\sqrt{b(b+c)^2}} + \frac{b}{\sqrt{c(c+a)^2}} + \frac{c}{\sqrt{a(a+b)^2}} \geq \frac{9}{4(ab+bc+ca)}.$$

8. Find the minimum value of the expression:

$$E(x) = |x-1| + |x-2| + \cdots + |x-100|,$$

where x is a real number.

9. Let n be an odd positive integer and let $x_1, x_2, \ldots, x_n$ be distinct real numbers. Find all one-to-one functions

$$f : \{x_1, x_2, \ldots, x_n\} \to \{x_1, x_2, \ldots, x_n\}$$

such that

$$|f(x_1) - x_1| = |f(x_2) - x_2| = \cdots = |f(x_n) - x_n|.$$

10. . Find the positive solutions of the following system of equations:

$$\begin{cases} \dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = 8\left(y^4 - x^4\right), \\ ax - by = x^4 - y^4 \end{cases}$$

where $a, b > 0$ are parameters.

11. Let $a, b, c > 0$. Solve the system of equations

$$\begin{cases} ax - by + \dfrac{1}{xy} = c, \\ bz - cx + \dfrac{1}{zx} = a, \\ cy - az + \dfrac{1}{yz} = b. \end{cases}$$

12. Consider the sequence

$$a_n = 2 - \frac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}, \quad n \geq 1.$$

Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \cdots + \sqrt{a_{119}}$$

is an integer.