Putnam Problem Session

September 18, 2013

1. Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose $I + A + A^2 + ... + A^{100} = 0$ (where I denotes the identity $d \times d$ matrix, so I has 1's on the main diagonal). Determine the positive integers $n \leq 100$ for which $An + A^{n+1} + ... + A^{100}$ has determinant ± 1.

2. Solve in real numbers the equation $3x - x^3 = \sqrt{x + 2}$.

3. Find

$$\sum_{k=1}^{\infty} \frac{k^2 - 2}{(k+2)!}.$$

4. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms and let $b_n = \frac{1}{na_n^2}$ for $n \geq 1$. Prove that

$$\sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + ... + b_n}$$

is convergent.

5. Let $\mathbb{C}$ denote the complex numbers and let $M_3(\mathbb{C})$ denote the 3×3 matrices with entries in $\mathbb{C}$. Suppose $A, B \in M_3(\mathbb{C}), B \neq 0$, and $AB = 0$. Prove that there exists $0 \neq D \in M_3(\mathbb{C})$ such that $AD = DA = 0$.

6. Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square.

7. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

8. Determine all invertible 2×2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A^T$, where A^T denotes the transpose of A.